MEN OF MATHEMATICS

by

E. T. BELL

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WITHOUT A MASS OF FOOTNOTES it would be impossible to cite authority for every statement of historical fact in the following pages. But little of the material consulted is available outside of large university libraries, and most of it is in foreign languages. For the principal dates and leading facts in the life of a particular man I have consulted the obituary notices (of the moderns); these are found in the proceedings of the learned societies of which the man in question was a member. Other details of interest are given in the correspondence between mathematicians and in their collected works. In addition to the few specific sources cited presently, bibliographies and references in the following have been especially helpful.

(1) The numerous historical notes and papers abstracted in the *Jahrbuch über die Fortschritte der Mathematik* (section on history of mathematics).

(2) The same in *Bibliotheca Mathematica.*

Only three of the sources are sufficiently "private" to need explicit citation. The life of Galois is based on the classic account by P. Dupuy in the *Annales scientifiques de l'École normale supérieure* (3^{me} série, tome 13, 1896), and the edited notes by Jules Tannery. The correspondence between Weierstrass and Sonja Kowalewski was published by Mittag-Leffler in the *Acta Mathematica* (also partly in the *Comptes rendus du 2^{me} Congrès international des Mathématiciens*, Paris, 1902). Many of the details concerning Gauss are taken from the book by W. Sartorius von Waltershausen, *Gauss zum Gedächtniss*, Leipzig, 1856.

It would be rash to claim that every date or spelling of proper names in the book is correct. Dates are used chiefly with the purpose of orienting the reader as to a man's age when he made his most original inventions. As to spellings, I confess my helplessness in the face of such variants as Basle, Bâle, Basel for one Swiss town, or Utzendorff, Uitzisdorf for another, each preferred by some admittedly reputable authority. When it comes to choosing between James and Johann, or between Wolfgang and Farkas, I take the easier way and identify the man otherwise.

Most of the portraits are reproduced from those in the David Eugene Smith Collection, Columbia University. The portrait of Newton is from an original mezzotint loaned by Professor E. C. Watson. The drawings have been constructed accurately by Mr. Eugene Edwards.

As on a previous occasion (*The Search for Truth*), it gives me great pleasure to thank Doctor Edwin Hubble and his wife, Grace, for their invaluable assistance. While I alone am responsible for all statements in the book, nevertheless it was a great help to have scholarly criticism (even if I did not always profit by it) from two experts in fields in which I cannot claim to be expert, and I trust that their constructive criticisms have lightened my own deficiencies. Doctor Morgan Ward also has criticized certain of the chapters and has made many helpful suggestions on matters in which he is expert. Toby, as before, has contributed much; in acknowledgment for what she has given, I have dedicated the book to her (if she will have it)—it is as much hers as mine.

Last, I wish to thank the staffs of the various libraries which have generously helped with the loan of rare books and bibliographical material. In particular I should like to thank the librarians at Stanford University, the University of California, the University of Chicago, Harvard University, Brown University, Princeton University, Yale University, The John Crerar Library (Chicago), and the California Institute of Technology.

E. T. Bell

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THEY SAY, WHAT THEY SAY, LET THEM SAY

(*Motto of Marischal College, Aberdeen*)

The science of Pure Mathematics, in its modern developments, may claim to be the most original creation of the human spirit.—A. N. WHITEHEAD (*Science and the Modern World*, 1925)

A mathematical truth is neither simple nor complicated in itself, it is.—ÉMILE LEMOINE

A mathematician who is not also something of a poet will never be a complete mathematician.—KARL WEIERSTRASS

I have heard myself accused of being an opponent, an enemy of mathematics, which no one can value more highly than I, for it accomplishes the very thing whose achievement has been denied me.—GOETHE

Mathematicians are like lovers. . . . Grant a mathematician the least principle, and he will draw from it a consequence which you must also grant him, and from this consequence another.—FONTENELLE

It is easier to square the circle than to get round a mathematician.—AUGUSTUS DE MORGAN

I regret that it has been necessary for me in this lecture to administer such a large dose of four-dimensional geometry. I do not apologise, because I am really not responsible for the fact that nature in its most fundamental aspect is four-dimensional. Things are what they are. . . .—A. N. WHITEHEAD (*The Concept of Nature*, 1920)

* * *

Number rules the universe.—THE PYTHAGOREANS

Mathematics is the Queen of the Sciences, and Arithmetic the Queen of Mathematics.—C. F. GAUSS

Thus number may be said to rule the whole world of quantity, and the four rules of arithmetic may be regarded as the complete equipment of the mathematician.—JAMES CLERK MAXWELL

The different branches of Arithmetic—Ambition, Distraction, Uglification, and Derision.—THE MOCK TURTLE (*Alice in Wonderland*)

God made the integers, all the rest is the work of man.—LEOPOLD KRONECKER

[Arithmetic] is one of the oldest branches, perhaps the very oldest branch, of human knowledge; and yet some of its most abstruse secrets lie close to its tritest truths.—H. J. S. SMITH

* * *

Plato's writings do not convince any mathematician that their author was strongly addicted to geometry. . . . We know that he encouraged mathematics. . . . But if—which nobody believes—the μηδεὶς ἀγεωμέτρητος εἰσίτω [Let no man ignorant of geometry enter] of Tzetzes had been written over his gate, it would no more have indicated the geometry within than a warning not to forget to bring a packet of sandwiches would now give promise of a good dinner.—AUGUSTUS DE MORGAN

There is no royal road to geometry.—MENAECHMUS (*to* ALEXANDER THE GREAT)

* * *

He studied and nearly mastered the six books of Euclid since he was a member of Congress.

He began a course of rigid mental discipline with the intent to improve his faculties, especially his powers of logic and language. Hence his fondness for Euclid, which he carried with him on the circuit till he could demonstrate with ease all the propositions in the six books; often studying far into the night, with a candle near his pillow, while his fellow-lawyers, half a dozen in a room, filled the air with interminable snoring.—ABRAHAM LINCOLN (*Short Autobiography*, 1860)

* * *

Strange as it may sound, the power of mathematics rests on its evasion of all unnecessary thought and on its wonderful saving of mental operations. —ERNST MACH

A single curve, drawn in the manner of the curve of prices of cotton, describes all that the ear can possibly hear as the result of the most complicated musical performance. . . . That to my mind is a wonderful proof of the potency of mathematics.—LORD KELVIN

* * *

The mathematician, carried along on his flood of symbols, dealing apparently with purely formal truths, may still reach results of endless importance for our description of the physical universe.—KARL PEARSON

Examples . . . which might be multiplied *ad libitum,* show how difficult it often is for an experimenter to interpret his results without the aid of mathematics.—LORD RAYLEIGH

But there is another reason for the high repute of mathematics: it is mathematics that offers the exact natural sciences a certain measure of security which, without mathematics, they could not attain.—ALBERT EINSTEIN

Mathematics is the tool specially suited for dealing with abstract concepts of any kind and there is no limit to its power in this field. For this reason a book on the new physics, if not purely descriptive of experimental work, must be essentially mathematical.—P. A. M. DIRAC (*Quantum Mechanics*, 1930)

As I proceeded with the study of Faraday, I perceived that his method of conceiving the phenomena [of electromagnetism] was also a mathematical one, though not exhibited in the conventional form of mathematical symbols. I also found that these methods were capable of being expressed in the ordinary mathematical forms, and thus compared with those of the professed mathematicians.—JAMES CLERK MAXWELL (*A Treatise on Electricity and Magnetism*, 1873)

* * *

Query 64. . . . Whether mathematicians . . . have not their mysteries, and, what is more, their repugnances and contradictions?—BISHOP BERKELEY

To create a healthy philosophy you should renounce metaphysics but be a good mathematician.—BERTRAND RUSSELL (*in a lecture*, 1935)

Mathematics is the only good metaphysics.—LORD KELVIN

How can it be that mathematics, being after all a product of human thought independent of experience, is so admirably adapted to the objects of reality?—ALBERT EINSTEIN (1920)

Every *new* body of discovery is mathematical in form, because there is no other guidance we can have.—C. G. DARWIN (1931)

The infinite! No other question has ever moved so profoundly the spirit of man.—DAVID HILBERT (1921)

The notion of infinity is our greatest friend; it is also the greatest enemy of our peace of mind. . . . Weierstrass taught us to believe that we had at last thoroughly tamed and domesticated this unruly element. Such however is not the case; it has broken loose again. Hilbert and Brouwer have set out to tame it once more. For how long? We wonder.—JAMES PIERPONT (*Bulletin of the American Mathematical Society*, 1928)

In my opinion a mathematician, in so far as he is a mathematician, need not preoccupy himself with philosophy—an opinion, moreover, which has been expressed by many philosophers.—HENRI LEBESGUE (1936)

God ever geometrizes.—PLATO

God ever arithmetizes.—C. G. J. JACOBI

The Great Architect of the Universe now begins to appear as a pure mathematician.—J. H. JEANS (*The Mysterious Universe*, 1930)

Mathematics is the most exact science, and its conclusions are capable of absolute proof. But this is so only because mathematics does not *attempt* to draw absolute conclusions. All mathematical truths are relative, conditional. —CHARLES PROTEUS STEINMETZ (1923)

It is a safe rule to apply that, when a mathematical or philosophical author writes with a misty profundity, he is talking nonsense.—A. N. WHITEHEAD (1911)

CHAPTER ONE

Introduction

THIS SECTION IS HEADED *Introduction* rather than *Preface* (which it really is) in the hope of decoying habitual preface-skippers into reading—for their own comfort—at least the following paragraphs down to the first row of stars before going on to meet some of the great mathematicians. I should like to emphasize first that this book is not intended, in any sense, to be a history of mathematics, or any section of such a history.

The lives of mathematicians presented here are addressed to the general reader and to others who may wish to see what sort of human beings the men were who created *modern* mathematics. Our object is to lead up to some of the dominating ideas governing vast tracts of mathematics as it exists today and to do this through the lives of the men responsible for those ideas.

Two criteria have been applied in selecting names for inclusion: the importance for modern mathematics of a man's work; the human appeal of the man's life and character. Some qualify under both heads, for example Pascal, Abel, and Galois; others, like Gauss and Cayley, chiefly under the first, although both had interesting lives. When these criteria clash or overlap in the case of several claimants to remembrance for a particular advance, the second has been given precedence as we are primarily interested here in mathematicians as human beings.

Of recent years there has been a tremendous surge of general interest in science, particularly physical science, and its bearing on our rapidly changing philosophical outlook on the universe. Numerous excellent accounts of current advances in science, written in as untechnical language as possible, have served to lessen the gap between the professional scientist and those who must make their livings at something other than science. In many of these expositions, especially those concerned with relativity and the modern quantum theory, names occur with which the general reader cannot be expected to be familiar—Gauss, Cayley, Riemann, and Hermite, for instance. With a knowledge of who these men were, their part in preparing for the explosive growth of physical science since 1900, and an appreciation of their rich personalities, the magnificent achievements of science fall into a truer perspective and take on a new significance.

The great mathematicians have played a part in the evolution of scientific and philosophic thought comparable to that of the philosophers and scientists themselves. To portray the leading features of that part through the lives of master mathematicians, presented against a background of some of the dominant problems of their times, is the purpose of the following chapters. The emphasis is wholly on modern mathematics, that is, on those great and simple guiding ideas of mathematical thought that are still of vital importance in living, creative science and mathematics.

It must not be imagined that the sole function of mathematics—"the handmaiden of the sciences"—is to serve science. Mathematics has also been called "the Queen of the Sciences." If occasionally the Queen has seemed to beg from the sciences she has been a very proud sort of beggar, neither asking nor accepting favors from any of her more affluent sister sciences. What she gets she pays for. Mathematics has a light and wisdom of its own, above any possible application to science, and it will richly reward any intelligent human being to catch a glimpse of what mathematics means to itself. This is not the old doctrine of art for art's sake; it is art for humanity's sake. After all, the whole purpose of science is not technology—God knows we have gadgets enough already; science also explores depths of a universe that will never, by any stretch of the imagination, be visited by human beings or affect our material existence. So we shall attend also to some of the things which the great mathematicians have considered worthy of loving understanding for their intrinsic beauty.

Plato is said to have inscribed "Let no man ignorant of geometry enter here" above the entrance to his Academy. No similar warning need be posted here, but a word of advice may save some over-conscientious reader unnecessary anguish. The gist of the story is in the lives and personalities of the creators of modern mathematics, not in the handful of formulas and diagrams scattered through the text. The basic ideas of modern mathematics, from which the whole vast and intricate complexity has been woven by thousands of workers, are simple, of boundless scope, and well within the understanding of any human being with normal intelligence. Lagrange (whom we shall meet later) believed that a mathematician has not thoroughly understood his own work till he has made it so clear that he can go out and explain it effectively to the first man he meets on the street.

This of course is an ideal and not always attainable. But it may be recalled that only a few years before Lagrange said this the Newtonian "law" of gravitation was an incomprehensible mystery to even highly educated persons. Yesterday the Newtonian "law" was a commonplace which every educated person accepted as simple and true; today Einstein's relativistic theory of gravitation is where Newton's "law" was in the early decades of the eighteenth century; tomorrow or the day after Einstein's theory will seem as "natural" as Newton's "law" seemed yesterday. With the help of time Lagrange's ideal is not unattainable.

Another great French mathematician, conscious of his own difficulties no less than his readers', counselled the conscientious not to linger too long over anything hard but to "Go on, and faith will come to you." In brief, if occasionally a formula, a diagram, or a paragraph seems too technical, skip it. There is ample in what remains.

Students of mathematics are familiar with the phenomenon of "slow development," or subconscious assimilation: the first time something new is studied the details seem too numerous and hopelessly confused, and no coherent impression of the whole is left on the mind. Then, on returning after a rest, it is found that everything has fallen into place with its proper emphasis—like the development of a photographic film. The majority of those who attack analytic geometry seriously for the first time experience something of the sort. The calculus on the other hand, with its aims clearly stated from the beginning, is usually grasped quickly. Even professional mathematicians often skim the work of others to gain a broad, comprehensive view of the whole before concentrating on the details of interest to them. Skipping is not a vice, as some of us were told by our puritan teachers, but a virtue of common sense.

As to the amount of mathematical knowledge necessary to understand *everything* that some will wisely skip, I believe it may be said honestly that a high school course in mathematics is sufficient. Matters far beyond such a course are frequently mentioned, but wherever they are, enough description has been given to enable anyone with high school mathematics to follow. For some of the most important ideas discussed in connection with their originators—groups, space of many dimensions, non-Euclidean geometry, and symbolic logic, for example —*less* than a high school course is ample for an understanding of the basic concepts. All that is needed is interest and an undistracted head. Assimilation of some of these invigorating ideas of modern mathematical thought will be found as refreshing as a drink of cold water on a hot day and as inspiring as any art.

To facilitate the reading, important definitions have been repeated where necessary, and frequent references to earlier chapters have been included from time to time.

The chapters need not be read consecutively. In fact, those with a speculative or philosophical turn of mind may prefer to read the last chapter first. With a few trivial displacements to fit the social background the chapters follow the chronological order.

It would be impossible to describe *all* the work of even the least prolific of the men considered, nor would it be profitable in an account for the general reader to attempt to do so. Moreover, much of the work of even the greater mathematicians of the past is now of only historical interest, having been included in more general points of view. Accordingly only some of the conspicuously new things each man did are described, and these have been selected for their originality and importance in modern thought.

Of the topics selected for description we may mention the following (among others) as likely to interest the general reader: the modern doctrine of the infinite (chapters 2, 29); the origin of mathematical probability (chapter 5); the concept and importance of a group (chapter 15); the meanings of invariance (chapter 21); non-Euclidean geometry (chapter 16 and part of 14); the origin of the mathematics of general relativity (last part of chapter 26); properties of the common whole numbers (chapter 4), and their modern generalization (chapter 25); the meaning and usefulness of so-called imaginary numbers—like $\sqrt{-1}$ (chapters 14, 19); symbolic reasoning (chapter 23). But anyone who wishes to get a glimpse of the power of the mathematical method, especially as applied to science, will be repaid by seeing what the calculus is about (chapters 2, 6).

Modern mathematics began with two great advances, analytic geometry and the calculus. The former took definite shape in 1637, the latter about 1666, although it did not become public property till a decade later. Though the idea behind it all is childishly simple, yet the method of analytic geometry is so powerful that very ordinary boys of seventeen can use it to prove results which would have baffled the greatest of the Greek geometers—Euclid, Archimedes, and Apollonius. The man, Descartes, who finally crystallized this great method had a particularly full and interesting life.

In saying that Descartes was responsible for the creation of analytic geometry we do not mean to imply that the new method sprang full-armed from his mind alone. Many before him had made significant advances toward the new method, but it remained for Descartes to take the final step and actually to put out the method as a definitely workable engine of geometrical proof, discovery, and invention. But even Descartes must share the honor with Fermat.

Similar remarks apply to most of the other advances of modern mathematics. A new concept may be "in the air" for generations until some one man—occasionally two or three together—sees clearly the essential detail that his predecessors missed, and the new thing comes into being. Relativity, for example, is sometimes said to have been the great invention reserved by time for the genius of Minkowski. The fact is, however, that Minkowski did not create the theory of relativity and that Einstein did. It seems rather meaningless to say that So-and-so might have done this or that if circumstances had been other than they were. Any one of us no doubt could jump over the moon if we and the physical universe were different from what we and it are, but the truth is that we do not make the jump.

In other instances, however, the credit for some great advance is not always justly placed, and the man who first used a new method more powerfully than its inventor sometimes gets more than his due. This seems to be the case, for instance, in the highly important matter of the calculus. Archimedes had the fundamental notion of limiting sums from which the integral calculus springs, and he not only had the notion but showed that he could apply it. Archimedes also used the method of the differential calculus in one of his problems. As we approach Newton and Leibniz in the seventeenth century the history of the calculus becomes extremely involved. The new method was more than merely "in the air" before Newton and Leibniz brought it down to earth; Fermat actually had it. He also invented the method of Cartesian geometry independently of Descartes. In spite of indubitable facts such as these we shall follow tradition and ascribe to each great leader what a majority vote says he should have, even at the risk of giving him a little more than his just due. Priority after all gradually loses its irritating importance as we recede in time from the men to whom it was a hotly contested cause of verbal battles while they and their partisans lived.

* * *

Those who have never known a professional mathematician may be rather surprised on meeting some, for mathematicians as a class are probably less familiar to the general reader than any other group of brain workers. The mathematician is a much rarer character in fiction than his cousin the scientist, and when he does appear in the pages of a novel or on the screen he is only too apt to be a slovenly dreamer totally devoid of common sense—comic relief. What sort of mortal is he in real life? Only by seeing in detail what manner of men some of the *great* mathematicians were and what kind of lives they lived, can we recognize the ludicrous untruth of the traditional portrait of a mathematician.

Strange as it may seem, not all of the great mathematicians have been professors in colleges or universities. Quite a few were soldiers by profession; others went into mathematics from theology, the law, and medicine, and one of the greatest was as crooked a diplomat as ever lied for the good of his country. A few have had no profession at all. Stranger yet, not all professors of mathematics have been mathematicians. But this should not surprise us when we think of the gulf between the average professor of poetry drawing a comfortable salary and the poet starving to death in his garret.

The lives that follow will at least suggest that a mathematician can be as human as anybody else—sometimes distressingly more so. In ordinary social contacts the majority have been normal. There have been eccentrics in mathematics, of course; but the percentage is no higher than in commerce or the professions. As a group the great mathematicians have been men of all-round ability, vigorous, alert, keenly interested in many things outside of mathematics and, in a fight, men with their full share of backbone. As a rule mathematicians have been bad customers to persecute; they have usually been capable of returning what they received with compound interest. For the rest they were geniuses of tremendous accomplishment marked off from the majority of their gifted fellowmen only by an irresistible impulse to do mathematics. On occasion mathematicians have been (and some still are in France) extremely able administrators.

In their politics the great mathematicians have ranged over the whole spectrum from reactionary conservatism to radical liberalism. It is probably correct to say that as a class they have tended slightly to the left in their political opinions. Their religious beliefs have included everything from the narrowest orthodoxy—sometimes shading into the blackest bigotry—to complete skepticism. A few were dogmatic and positive in their assertions concerning things about which they knew nothing, but most have tended to echo the great Lagrange's "I do not know."

Another characteristic calls for mention here, as several writers and artists (some from Hollywood) have asked that it be treated—the sex life of great mathematicians. In particular these inquirers wish to know how many of the great mathematicians have been perverts—a somewhat indelicate question, possibly, but legitimate enough to merit a serious answer in these times of preoccupation with such topics. None. Some lived celibate lives, usually on account of economic disabilities, but the majority were happily married and brought up their children in a civilized, intelligent manner. The children, it may be noted in passing, were often gifted far above the average. A few of the great mathematicians of bygone centuries kept mistresses when such was the fashionable custom of their times. The only mathematician discussed here whose life might offer something of interest to a Freudian is Pascal.

Returning for a moment to the movie ideal of a mathematician, we note that sloppy clothes have not been the invariable attire of great mathematicians. All through the long history of mathematics about which we have fairly detailed knowledge, mathematicians have paid the same amount of attention to their personal appearance as any other equally numerous group of men. Some have been fops, others slovens; the majority, decently inconspicuous. If today some earnest individual affecting spectacular clothes, long hair, a black sombrero, or any other mark of exhibitionism, assures you that he is a mathematician, you may safely wager that he is a psychologist turned numerologist.

The psychological peculiarities of great mathematicians is another topic in which there is considerable interest. Poincaré will tell us something about the psychology of mathematical creation in a later chapter. But on the general question not much can be said till psychologists call a truce and agree among themselves as to what is what. On the whole the great mathematicians have lived richer, more virile lives than those that fall to the lot of the ordinary hard-working mortal. Nor has this richness been wholly on the side of intellectual adventuresomeness. Several of the greater mathematicians have had more than their share of physical danger and excitement, and some of them have been implacable haters—or, what is ultimately the same, expert controversialists. Many have known the lust of battle in their prime, reprehensibly enough, no doubt, but still humanly enough, and in knowing it they have experienced something no jellyfish has ever felt: "Damn braces, Bless relaxes," as that devout Christian William Blake put it in his *Proverbs of Hell.*

This brings us to what at first sight (from the conduct of several of the men considered here) may seem like a significant trait of mathematicians—their hair-trigger quarrelsomeness. Following the lives of several of these men we get the impression that a great mathematician is more likely than not to think others are stealing his work, or disparaging it, or not doing him sufficient honor, and to start a row to recover imaginary rights. Men who should have been above such brawls seem to have gone out of their way to court battles over priority in discovery and to accuse their competitors of plagiarism. We shall see enough dishonesty to discount the superstition that the pursuit of truth necessarily makes a man truthful, but we shall not find indubitable evidence that mathematics makes a man bad-tempered and quarrelsome.

Another "psychological" detail of a similar sort is more disturbing. Envy is carried up to a higher level. Narrow nationalism and international jealousies, even in impersonal pure mathematics, have marred the history of discovery and invention to such an extent that it is almost impossible in some important instances to get at the facts or to form a just estimate of the significance of a particular man's work for modern thought. Racial fanaticism—especially in recent times—has also complicated the task of anyone who may attempt to give an unbiased account of the lives and work of scientific men outside his own race or nation.

An impartial account of western mathematics, including the award to each man and to each nation of its just share in the intricate development, could be written only by a Chinese historian. He alone would have the patience and the detached cynicism necessary for disentangling the curiously perverted pattern to discover whatever truth may be concealed in our variegated occidental boasting.

* * *

Even in restricting our attention to the modern phase of mathematics we are faced with a problem of selection that must be solved somehow. Before the solution adopted here is indicated it will be of interest to estimate the amount of labor that would be required for a detailed history of mathematics on a scale similar to that of a political history for any important epoch, say that of the French Revolution or the American Civil War.

When we begin unravelling a particular thread in the history of mathematics we soon get a discouraged feeling that mathematics itself is like a vast necropolis to which constant additions are being made for the eternal preservation of the newly dead. The recent arrivals, like some of the few who were shelved for perpetual remembrance 5000 years ago, must be so displayed that they shall seem to retain the full vigor of the manhood in which they died; in fact the illusion must be created that they have not yet ceased living. And the deception must be so natural that even the most skeptical archaeologist prowling through the mausoleums shall be moved to exclaim with living mathematicians themselves that mathematical truths are immortal, imperishable; the same yesterday, today, and forever; the very stuff of which eternal verities are fashioned and the one glimpse of changelessness behind all the recurrent cycles of birth, death, and decay our race has ever caught. Such may indeed be the fact; many, especially those of the older generation of mathematicians, hold it to be no less.

But the mere spectator of mathematical history is soon overwhelmed by the appalling mass of mathematical inventions that still maintain their vitality and importance for modern work, as discoveries of the past in any other field of scientific endeavor do not, after centuries and tens of centuries.

A span of less than a hundred years covers everything of significance in the French Revolution or the American Civil War, and less than five hundred leaders in either played parts sufficiently memorable to merit recording. But the army of those who have made at least one definite contribution to mathematics as we know it soon becomes a mob as we look back over history; 6000 or 8000 names press forward for some word from us to preserve them from oblivion, and once the bolder leaders have been recognized it becomes largely a matter of arbitrary, illogical legislation to judge who of the clamoring multitude shall be permitted to survive and who be condemned to be forgotten.

This problem scarcely presents itself in describing the development of the physical sciences. They also reach far back into antiquity; yet for the most of them 350 years is a sufficient span to cover everything of importance to modern thought. But whoever attempts to do full, human justice to mathematics and mathematicians will have a wilderness of 6000 years in which to exercise such talents as he may have, with that mob of 6000 to 8000 claimants before him for discrimination and attempted justice.

The problem becomes more desperate as we approach our own times. This is by no means due to our closer proximity to the men of the two centuries immediately preceding our own, but to the universally acknowledged fact (among professional mathematicians) that the nineteenth century, prolonged into the twentieth, was, and is, the greatest age of mathematics the world has ever known. Compared to what glorious Greece did in mathematics the nineteenth century is a bonfire beside a penny candle.

What threads shall we follow to guide us through this labyrinth of mathematical inventions? The main thread has already been indicated: that which leads from the half-forgotten past to some of those dominating concepts which now govern boundless empires of mathematics—but which may themselves be dethroned tomorrow to make room for yet vaster generalizations. Following this main thread we shall pass by the *developers* in favor of the *originators.*

Both inventors and perfectors are necessary to the progress of any science. Every explorer must have, in addition to his scouts, his followers to inform the world as to what he has discovered. But to the majority of human beings, whether justly or not is beside the point, the explorer who first shows the new way is the more arresting personality, even if he himself stumbles forward but half a step. We shall follow the originators in preference to the developers. Fortunately for historical justice the majority of the great originators in mathematics have also been peerless developers.

Even with this restriction the path from the past to the present may not always be clear to those who have not already followed it. So we may state here briefly what the main guiding clue through the whole history of mathematics is.

From the earliest times two opposing tendencies, sometimes helping one another, have governed the whole involved development of mathematics. Roughly these are the *discrete* and the *continuous.*

The discrete struggles to describe all nature and all mathematics atomistically, in terms of distinct, recognizable individual elements, like the bricks in a wall, or the numbers 1,2,3, . . . The continuous seeks to apprehend natural phenomena—the course of a planet in its orbit, the flow of a current of electricity, the rise and fall of the tides, and a multitude of other appearances which delude us into believing that we know nature—in the mystical formula of Heraclitus: "All things flow." Today (as will be seen in the concluding chapter), "flow," or its equivalent, "continuity," is so unclear as to be almost devoid of meaning. However, let this pass for the moment.

Intuitively we *feel* that we *know* what is meant by "continuous motion"—as of a bird or a bullet through the air, or the fall of a raindrop. The motion is *smooth*; it *does not proceed by jerks*; it is *unbroken.* In *continuous* motion or, more generally, in the concept of continuity itself, the *individualized* numbers 1,2,3, . . . , are *not* the appropriate mathematical image. *All* the points on a segment of a straight line, for instance, have no such clear-cut individualities as have the numbers of the sequence 1,2,3, . . . , where *the step from one member of the sequence to the next is the same* (namely 1: $1 + 2 = 3$, $1 + 3 = 4$, and so on); for *between* any two points on the line segment, no matter how close together the points may be, we can always *find*, or at least *imagine*, *another* point: *there is no "shortest" step from one point to the "next."* In fact there is no *next* point at all.

The last—the conception of *continuity*, "no nextness"—when developed in the manner of Newton, Leibniz, and their successors leads out into the boundless domain of *the calculus* and its innumerable applications to science and technology, and to all that is today called *mathematical analysis.* The other, the *discrete* pattern based on 1,2,3, . . . , is the domain of algebra, the theory of numbers, and symbolic logic. Geometry partakes of both the continuous and the discrete.

A major task of mathematics today is to harmonize the continuous and the discrete, to include them in one comprehensive mathematics, and to eliminate obscurity from both.

* * *

It may be doing our predecessors an injustice to emphasize modern mathematical thought with but little reference to the pioneers who took the first and possibly the most difficult steps. But nearly everything useful that was done in mathematics before the seventeenth century has suffered one of two fates: either it has been so greatly simplified that it is now part of every regular school course, or it was long since absorbed as a detail in work of greater generality.

Things that now seem as simple as common sense—our way of writing numbers, for instance, with its "place system" of value and the introduction of a symbol for zero, which put the essential finishing touch to the place system—cost incredible labor to invent. Even simpler things, containing the very essence of mathematical thought —*abstractness and generality*, must have cost centuries of struggle to devise; yet their originators have vanished leaving not a trace of their lives and personalities. For example, as Bertrand Russell observed, "It must have taken many ages to discover that a brace of pheasants and a couple of days were both instances of the number two." And it took some twenty five centuries of *civilization* to evolve Russell's own logical definition of "two" or of any cardinal number (reported in the concluding chapter).

Again, the conception of a point, which we (erroneously) think we fully understand when we begin school geometry must have come very late in man's career as an artistic, cave-painting animal. Horace Lamb, an English mathematical physicist, would "erect a monument to the unknown mathematical inventor of the mathematical point as the supreme type of that abstraction which has been a necessary condition of scientific work from the beginning."

Who, by the way, *did* invent the mathematical point? In one sense Lamb's forgotten man; in another, Euclid with his definition "a point is that which has no parts and which has no magnitude"; in yet a third sense Descartes with his invention of the "coordinates of a point": until finally in geometry as experts practise it today the mysterious "point" has joined the forgotten man and all his gods in everlasting oblivion, to be replaced by something more usable—*a set of numbers written in a definite order.*

The last is a modern instance of the abstractness and precision toward which mathematics strives constantly, only to realize when abstractness and precision are attained that a higher degree of abstractness and a sharper precision are demanded for clear understanding. Our own conception of a "point" will no doubt evolve into something yet more abstract. Indeed the "numbers" in terms of which points are described today dissolved about the beginning of this century into the shimmering blue of pure logic, which in its turn seems about to vanish in something rarer and even less substantial.

It is not necessarily true then that a step-by-step following of our predecessors is the sure way to understand either their conception of mathematics or our own. Such a retracing of the path that has led up to our present outlook would undoubtedly be of great interest in itself. But it is quicker to glance back over the terrain from the hilltop on which we now stand. The false steps, the crooked trails, and the roads that led nowhere fade out in the distance, and only the broad highways are seen leading straight back to the past, where we lose them in the mists of uncertainty and conjecture. Neither space nor number, nor even time, have the same significance for us that they had for the men whose great figures appear dimly through the mist.

A Pythagorean of the sixth century before Christ could intone "Bless us, divine Number, thou who generatest gods and men"; a Kantian of the nineteenth century could refer confidently to "space" as a form of "pure intuition"; a mathematical astronomer could announce a decade ago that the Great Architect of the Universe is a pure mathematician. The most remarkable thing about all of these profound utterances is that human beings no stupider than ourselves once thought they made sense.

To a modern mathematician such all-embracing generalities mean less than nothing. Yet in parting with its claim to be the universal generator of gods and men mathematics has gained something more substantial, a faith in itself and in its ability to create human values.

Our point of view has changed—and is still changing. To Descartes' "Give me space and motion and I will give you a world," Einstein today might retort that altogether too much is being asked, and that the demand is in fact meaningless: without a "world"—matter—there is neither "space" nor "motion." And to quell the turbulent, muddled mysticism of Leibniz in the seventeenth century, over the mysterious $\sqrt{-1}$: "The Divine Spirit found a sublime out-let in that wonder of analysis, the portent of the ideal, that mean between being and not-being, which we call the imaginary [square] root of negative unity," Hamilton in the 1840's constructed a number-couple which any intelligent child can understand and manipulate, and which does for mathematics and science all that the misnamed "imaginary" ever did. The mystical "not-being" of the seventeenth century Leibniz is seen to have a "being" as simple as ABC.

Is this a loss? Or does a modern mathematician lose anything of value when he seeks through the postulational method to track down that elusive "feeling" described by Heinrich Hertz, the discoverer of wireless waves: "One cannot escape the feeling that these mathematical formulas have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than was originally put into them"?

Any competent mathematician will understand Hertz' feeling, but he will also incline to the belief that whereas continents and wireless waves are discovered, dynamos and mathematics are invented and do what we make them do. We can still dream but we need not deliberately court nightmares. If it is true, as Charles Darwin asserted, that "Mathematics seems to endow one with something like a new sense," that sense is the sublimated common sense which the physicist and engineer Lord Kelvin declared mathematics to be.

Is it not closer to our own habits of thought to agree temporarily with Galileo that "Nature's great book is written in mathematical symbols" and let it go at that, than to assert with Plato that "God ever geometrizes," or with Jacobi that "God ever arithmetizes"? If we care to inspect the symbols in nature's great book through the critical eyes of modern science we soon perceive that we ourselves did the writing, and that we used the particular script we did because we invented it to fit our own understanding. Some day we may find a more expressive shorthand than mathematics for correlating our experiences of the physical universe—unless we accept the creed of the scientific mystics that everything *is* mathematics and is not merely *described* for our convenience in mathematical language. *If* "Number rules the universe" as Pythagoras asserted, Number is merely our delegate to the throne, for we rule Number.

When a modern mathematician turns aside for a moment from his symbols to communicate to others the feeling that mathematics inspires in him, he does not echo Pythagoras and Jeans, but he may quote what Bertrand Russell said about a quarter of a century ago: "Mathematics, rightly viewed, possesses not only truth but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show."

Another, familiar with what has happened to our conception of mathematical "truth" in the years since Russell praised the beauty of mathematics, might refer to the "iron endurance" which some acquire from their attempt to understand what mathematics means, and quote James Thomson's lines (which close this book) in description of Dürer's *Melencolia* (the frontispiece). And if some devotee is reproached for spending his life on what to many may seem the selfish pursuit of a beauty having no immediate reflection in the lives of his fellowmen, he may repeat Poincaré's "Mathematics for mathematics' sake. People have been shocked by this formula and yet it is as good as life for life's sake, if life is but misery."

* * *

To form an estimate of what modern mathematics compared to ancient has accomplished, we may first look at the mere bulk of the work in the period after 1800 compared to that before 1800. The most extensive history of mathematics is that of Moritz Cantor, *Geschichte der Mathematik*, in three large closely printed volumes (a fourth, by collaborators, supplements the three). The four volumes total about 3600 pages. Only the outline of the development is given by Cantor; there is no attempt to go into details concerning the contributions described, nor are technical terms explained so that an outsider could understand what the whole story is about, and biography is cut to the bone; the history is addressed to those who have some technical training. This history *ends with the year 1799*—just before modern mathematics began to feel its freedom. What if the *outline* history of mathematics in the nineteenth century alone were attempted on a similar scale? It has been estimated that nineteen or twenty volumes the size of Cantor's would be required to tell the story, say about 17,000 pages. The nineteenth century, on this scale, contributed to mathematical knowledge about *five times as much* as was done in the whole of preceding history.

The beginningless period before 1800 breaks quite sharply into two. The break occurs about the year 1700, and is due mainly to Isaac Newton (1642–1727). Newton's greatest rival in mathematics was Leibniz (1646–1716). According to Leibniz, of all mathematics up to the time of Newton, the more important half is due to Newton. This estimate refers to the power of Newton's general methods rather than to the bulk of his work; the *Principia* is still rated as the most massive addition to scientific thought ever made by one man.

Continuing back into time beyond 1700 we find nothing comparable till we reach the Golden Age of Greece—a step of nearly 2000 years. Farther back than 600 B.C. we quickly pass into the shadows, coming out into the light again for a moment in ancient Egypt. Finally we arrive at the first great age of mathematics, about 2000 B.C., in the Euphrates Valley.

The descendants of the Sumerians in Babylon appear to have been the first "moderns" in mathematics; certainly their attack on algebraic equations is more in the spirit of the algebra we know than anything done by the Greeks in their Golden Age. More important than the technical algebra of these ancient Babylonians is their recognition—as shown by their work—of the necessity for *proof* in mathematics. Until recently it had been supposed that the Greeks were the first to recognize that proof is demanded for mathematical propositions. This was one of the most important steps ever taken by human beings. Unfortunately it was taken so long ago that it led nowhere in particular so far as our own civilization is concerned—unless the Greeks followed consciously, which they may well have done. They were not particularly generous to their predecessors.

Mathematics then has had four great ages: the Babylonian, the Greek, the Newtonian (to give the period around 1700 a name), and the recent, beginning about 1800 and continuing to the present day. Competent judges have called the last the Golden Age of Mathematics.

Today mathematical invention (discovery, if you prefer) is going forward more vigorously than ever. The only thing, apparently, that can stop its progress is a general collapse of what we have been pleased to call civilization. If that comes, mathematics may go underground for centuries, as it did after the decline of Babylon; but if history repeats itself, as it is said to do, we may count on the spring bursting forth again, fresher and clearer than ever, long after we and all our stupidities shall have been forgotten.

CHAPTER TWO

Modern Minds in Ancient Bodies

ZENO, EUDOXUS, ARCHIMEDES

. . . the glory that was Greece
And the grandeur that was Rome.
—E. A. POE

TO APPRECIATE our own Golden Age of mathematics we shall do well to have in mind a few of the great, simple guiding ideas of those whose genius prepared the way for us long ago, and we shall glance at the lives and works of three Greeks: Zeno (495–435 B.C.), Eudoxus (408–355 B.C.), and Archimedes (287–212 B.C.). Euclid will be noticed much later, where his best work comes into its own.

Zeno and Eudoxus are representative of two vigorous opposing schools of mathematical thought which flourish today, the critical-destructive and the critical-constructive. Both had minds as penetratingly critical as their successors in the nineteenth and twentieth centuries. This statement can of course be inverted: Kronecker (1823–1891) and Brouwer (1881–), the modern critics of mathematical analysis—the theories of the infinite and the continuous—are as ancient as Zeno; the creators of the modern theories of continuity and the infinite, Weierstrass (1815–1897), Dedekind (1831–1916), and Cantor (1845–1918) are intellectual contemporaries of Eudoxus.

Archimedes, the greatest intellect of antiquity, is modern to the core. He and Newton would have understood one another perfectly, and it is just possible that Archimedes, could he come to life long enough to take a post-graduate course in mathematics and physics, would understand Einstein, Bohr, Heisenberg, and Dirac better than they understand themselves. Of all the ancients Archimedes is the only one who habitually thought with the unfettered freedom that the greater mathematicians permit themselves today with all the hard-won gains of twenty five centuries to smooth their way, for he alone of all the Greeks had sufficient stature and strength to stride clear over the obstacles thrown in the path of mathematical progress by frightened geometers who had listened to the philosophers.

Any list of the three "greatest" mathematicians of all history would include the name of Archimedes. The other two usually associated with him are Newton (1642–1727) and Gauss (1777–1855). Some, considering the relative wealth—or poverty—of mathematics and physical science in the respective ages in which these giants lived, and estimating their achievements against the background of their times, would put Archimedes first. Had the Greek mathematicians and scientists followed Archimedes rather than Euclid, Plato, and Aristotle, they might easily have anticipated the age of modern mathematics, which began with Descartes (1596–1650) and Newton in the seventeenth century, and the age of modern physical science inaugurated by Galileo (1564–1642) in the same century, by two thousand years.

Behind all three of these precursors of the modern age looms the half-mythical figure of Pythagoras (569?–500? B.C.), mystic, mathematician, investigator of nature to the best of his self-hobbled ability, "one tenth of him genius, nine-tenths sheer fudge." His life has become a fable, rich with the incredible accretions of his prodigies; but only this much is of importance for the development of mathematics as distinguished from the bizarre number-mysticism in which he clothed his cosmic speculations: he travelled extensively in Egypt, learned much from the priests and believed more; visited Babylon and repeated his Egyptian experiences; founded a secret Brotherhood for high mathematical thinking and nonsensical physical, mental, moral, and ethical speculation at Croton in southern Italy; and, out of all this, made two of the greatest contributions to mathematics in its entire history. He died, according to one legend, in the flames of his own school fired by political and religious bigots who stirred up the masses to protest against the enlightenment which Pythagoras sought to bring them. *Sic transit gloria mundi.*

Before Pythagoras it had not been clearly realized that *proof* must proceed from *assumptions*. Pythagoras, according to persistent tradition, was the first European to insist that the *axioms*, the *postulates*, be set down first in developing geometry and that the entire development thereafter shall proceed by applications of close deductive reasoning to the axioms. Following current practice we shall use "postulate," instead of "axiom" hereafter, as "axiom" has a pernicious historical association of "self-evident, necessary truth" which "postulate" does not have; a postulate is an arbitrary assumption laid down by the mathematician himself and not by God Almighty.

Pythagoras then imported *proof* into mathematics. This is his greatest achievement. Before him geometry had been largely a collection of rules of thumb empirically arrived at without any clear indication of the mutual connections of the rules, and without the slightest suspicion that all were deducible from a comparatively small number of postulates. Proof is now so commonly taken for granted as the very spirit of mathematics that we find it difficult to imagine the primitive thing which must have preceded mathematical reasoning.

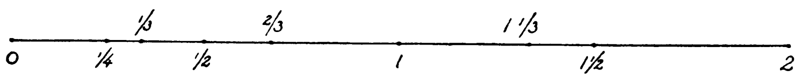
Pythagoras' second outstanding mathematical contribution brings us abreast of living problems. This was the discovery, which humiliated and devastated him, that the common whole numbers 1,2,3, . . . are insufficient for the construction of mathematics even in the rudimentary form in which he knew it. Before this capital discovery he had preached like an inspired prophet that all nature, the entire universe in fact, physical, metaphysical, mental, moral, mathematical—*everything*—is built on the *discrete* pattern of the integers 1,2,3, . . . and is interpretable in terms of these God-given bricks alone; God, he declared indeed, *is* "number," and by that he meant common whole number. A sublime conception, no doubt, and beautifully simple, but as unworkable as its echo in Plato—"God ever geometrizes," or in Jacobi—"God ever arithmetizes," or in Jeans—"The Great Architect of the Universe now begins to appear as a mathematician." One obstinate mathematical discrepancy demolished Pythagoras' discrete philosophy, mathematics, and metaphysics. But, unlike some of his successors, he finally accepted defeat—after struggling unsuccessfully to suppress the discovery which abolished his creed.

This was what knocked his theory flat: it is impossible to find two whole numbers such that the square of one of them is equal to twice the square of the other. This can be proved by a simple argument* within the reach of anyone who has had a few weeks of algebra,

*Let $a^2 = 2b^2$, where, without loss of generality, a, b are whole numbers without any common factor greater than 1 (such a factor could be cancelled from the assumed equation). If a is *odd*, we have an immediate contradiction, since $2b^2$ is *even*; if a is *even*, say $2c$, then $4c^2 = 2b^2$, or $2c^2 = b^2$, so b is *even*, and hence a, b have the common factor 2, again a contradiction.

or even by anyone who thoroughly understands elementary arithmetic. Actually Pythagoras found his stumbling-block in geometry: the ratio of the side of a square to one of its diagonals cannot be expressed as the ratio of any two whole numbers. This is equivalent to the statement above about squares of whole numbers. In another form we would say that the square root of 2 is *irrational,* that is, is not equal to any whole number or decimal fraction, or sum of the two, got by dividing one whole number by another. Thus even so simple a geometrical concept as that of the diagonal of a square defies the integers 1,2,3, . . . and negates the earlier Pythagorean philosophy. We can easily construct the diagonal *geometrically, but we cannot measure it in any finite number of steps.* This impossibility sharply and clearly brought irrational numbers and the infinite (non-terminating) processes which they seem to imply to the attention of mathematicians. Thus the square root of two can be calculated to any required *finite* number of decimal places by the process taught in school or by more powerful methods, but the decimal never "repeats" (as that for 1/7 does, for instance), nor does it ever terminate. In this discovery Pythagoras found the taproot of modern mathematical analysis.

Issues were raised by this simple problem which are not yet disposed of in a manner satisfactory to all mathematicians. These concern the mathematical concepts of the infinite (the unending, the uncountable), limits, and continuity, concepts which are at the root of modern analysis. Time after time the paradoxes and sophisms which crept into mathematics with these apparently indispensable concepts have been regarded as finally eliminated, only to reappear a generation or two later, changed but yet the same. We shall come across them, livelier than ever, in the mathematics of our time. The following is an extremely simple, intuitively obvious picture of the situation.



Consider a straight line two inches long, and imagine it to have been traced by the "continuous" "motion" of a "point." The words in quotes are those which conceal the difficulties. Without analysing them we easily persuade ourselves that we picture what they signify. Now label the left-hand end of the line 0 and the right-hand end 2.

Half-way between 0 and 2 we naturally put 1; half-way between 0 and 1 we put ½; half-way between 0 and ½ we put ¼, and so on. Similarly, between 1 and 2 we mark the place 1½, between 1½ and 2, the place 1¾, and so on. Having done this we may proceed in the same way to mark ⅓, ⅔, 1⅓, 1⅔, and then split each of the resulting segments into smaller equal segments. Finally, "in imagination," we can conceive of this process having been carried out for *all* the common fractions and common mixed numbers which are greater than 0 and less than 2; the conceptual division-points give us *all the rational numbers between 0 and 2*. There are an infinity of them. Do they completely "cover" the line? No. To what point does the square root of 2 correspond? No point, because this square root is not obtainable by dividing *any* whole number by another. But the square root of 2 is obviously a "number" of some sort;* its representative point lies somewhere between 1.41 and 1.42, and we can cage it down as closely as we please. To cover the line completely we are forced to imagine or to invent infinitely more "numbers" than the rationals. That is, if we accept the line as being *continuous*, and *postulate* that to each point of it corresponds one, and only one, "real number." The same kind of imagining can be carried on to the entire plane, and farther, but this is sufficient for the moment.

Simple problems such as these soon lead to very serious difficulties. With regard to these difficulties the Greeks were divided, just as we are, into two irreconcilable factions; one stopped dead in its mathematical tracks and refused to go on to analysis—the integral calculus, at which we shall glance when we come to it; the other attempted to overcome the difficulties and succeeded in convincing itself that it had done so. Those who stopped committed but few mistakes and were comparatively sterile of truth no less than of error; those who went on discovered much of the highest interest to mathematics and rational thought in general, some of which may be open to destructive criticism, however, precisely as has happened in our own generation. From the earliest times we meet these two distinct and antagonistic types of mind: the justifiably cautious who hang back because the ground quakes under their feet, and the bolder pioneers who leap the chasm to find treasure and comparative safety on the other side. We shall look first at one of those who refused to leap. For penetrating

* The inherent viciousness of such an assumption is obvious.

subtlety of thought we shall not meet his equal till we reach the twentieth century and encounter Brouwer.

Zeno of Elea (495–435 B.C.) was a friend of the philosopher Parmenides, who, when he visited Athens with his patron, shocked the philosophers out of their complacency by inventing four innocent paradoxes which they could not dissipate in words. Zeno is said to have been a self-taught country boy. Without attempting to decide what was his purpose in inventing his paradoxes—authorities hold widely divergent opinions—we shall merely state them. With these before us it will be fairly obvious that Zeno would have objected to our "infinitely continued" division of that two-inch line a moment ago. This will appear from the first two of his paradoxes, the *Dichotomy* and the *Achilles*. The last two, however, show that he would have objected with equal vehemence to the *opposite* hypothesis, namely that the line is *not* "infinitely divisible" but is composed of a *discrete* set of points that can be counted off 1,2,3, ... All four together constitute an iron wall beyond which progress appears to be impossible.

First, the *Dichotomy*. Motion is impossible, because whatever moves must reach the middle of its course *before* it reaches the end; but *before* it has reached the middle it must have reached the quarter-mark, and so on, *indefinitely*. Hence the motion can never even start.

Second, the *Achilles*. Achilles running to overtake a crawling tortoise ahead of him can never overtake it, because he must first reach the place from which the tortoise started; when Achilles reaches that place, the tortoise has departed and so is still ahead. Repeating the argument we easily see that the tortoise will always be ahead.

Now for the other side.

The *Arrow*. A moving arrow at any instant is either at rest or not at rest, that is, moving. If the instant is indivisible, the arrow cannot move, for if it did the instant would immediately be divided. But time is made up of instants. As the arrow cannot move in any one instant, it cannot move in any time. Hence it always remains at rest.

The *Stadium*. "To prove that half the time may be equal to double the time. Consider three rows of bodies

	First Position					*Second Position*					
(A)	o	o	o	o	(A)		o	o	o	o	
(B)	o	o	o	o	(B)	o	o	o	o		
(C)	o	o	o	o	(C)			o	o	o	o

one of which (A) is at rest while the other two (B), (C) are moving with equal velocities in opposite directions. By the time they are all in the same part of the course (B) will have passed twice as many of the bodies in (C) as in (A). Therefore the time which it takes to pass (A) is twice as long as the time it takes to pass (C). But the time which (B) and (C) take to reach the position of (A) is the same. Therefore double the time is equal to half the time." (Burnet's translation.) It is helpful to imagine (A) as a circular picket fence.

These, in non-mathematical language, are the sort of difficulties the early grapplers with continuity and infinity encountered. In books written twenty years or so ago it was said that "the positive theory of infinity" created by Cantor, and the like for "irrational" numbers, such as the square root of 2, invented by Eudoxus, Weierstrass, and Dedekind, had disposed of all these difficulties once and forever. Such a statement would not be accepted today by all schools of mathematical thought. So in dwelling upon Zeno we have in fact been discussing ourselves. Those who wish to see any more of him may consult Plato's *Parmenides.* We need remark only that Zeno finally lost his head for treason or something of the sort, and pass on to those who did not lose their heads over his arguments. Those who stayed behind with Zeno did comparatively little for the advancement of mathematics, although their successors have done much to shake its foundations.

Eudoxus (408–355 B.C.) of Cnidus inherited the mess which Zeno bequeathed the world and not much more. Like more than one man who has left his mark on mathematics, Eudoxus suffered from extreme poverty in his youth. Plato was in his prime while Eudoxus lived and Aristotle was about thirty when Eudoxus died. Both Plato and Aristotle, the leading philosophers of antiquity, were much concerned over the doubts which Zeno had injected into mathematical reasoning and which Eudoxus, in his theory of proportion—"the crown of Greek mathematics"—was to allay till the last quarter of the nineteenth century.

As a young man Eudoxus moved to Athens from Tarentum, where he had studied with Archytas (428–347 B.C.), a first-rate mathematician, administrator, and soldier. Arriving in Athens, Eudoxus soon fell in with Plato. Being too poor to live near the Academy, Eudoxus trudged back and forth every day from the Piraeus where fish and olive oil were cheap and lodging was to be had for a smile in the right place.

Although he himself was not a mathematician in the technical sense, Plato has been called "the maker of mathematicians," and it cannot be denied that he did irritate many infinitely better mathematicians than himself into creating some real mathematics. As we shall see, his total influence on the development of mathematics was probably baneful. But he did recognize what Eudoxus was and became his devoted friend until he began to exhibit something like jealousy toward his brilliant protégé. It is said that Plato and Eudoxus made a journey to Egypt together. If so, Eudoxus seems to have been less credulous than his predecessor Pythagoras; Plato however shows the effects of having swallowed vast quantities of the number-mysticism of the East. Finding himself unpopular in Athens, Eudoxus finally settled and taught at Cyzicus, where he spent his last years. He studied medicine and is said to have been a practising physician and legislator on top of his mathematics. As if all this were not enough to keep one man busy he undertook a serious study of astronomy, to which he made outstanding contributions. In his scientific outlook he was centuries ahead of his verbalizing, philosophizing contemporaries. Like Galileo and Newton he had a contempt for speculations about the physical universe which could not be checked by observation and experience. If by getting to the sun, he said, he could ascertain its shape, size, and nature, he would gladly share the fate of Phaëthon, but in the meantime he would not guess.

Some idea of what Eudoxus did can be seen from a very simple problem. To find the area of a rectangle we multiply the length by the breadth. Although this sounds intelligible it presents serious difficulties unless both sides are measurable by *rational* numbers. Passing these particular difficulties we see them in a more evident form in the next simplest type of problem, that of finding the length of a *curved* line, or the area of a *curved* surface, or the volume enclosed by *curved* surfaces.

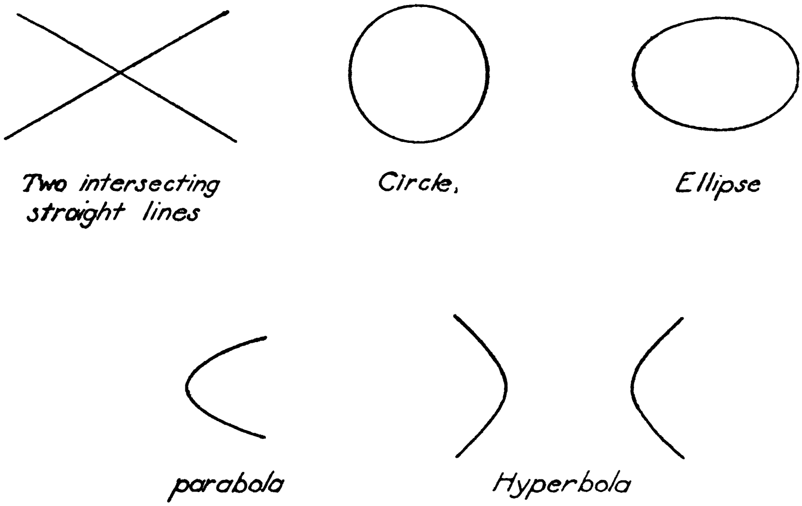
Any young genius wishing to test his mathematical powers may try to devise a method for doing these things. Provided he has never seen it done in school, how would he proceed to give a rigorous proof of the formula for the circumference of a circle of any given radius? Whoever does that entirely on his own initiative may justly claim to be a mathematician of the first rank. The moment we pass from figures bounded by *straight* lines or *flat* surfaces we run slap into all the problems of continuity, the riddles of the infinite and the mazes of irrational numbers. Eudoxus devised the first logically satisfactory method, which Euclid reproduced in Book V of his *Elements*, for handling such problems. In his *method of exhaustion*, applied to the computation of areas and volumes, Eudoxus showed that we need not assume the "existence" of "infinitely small quantities." It is sufficient for the purposes of mathematics to be able to reach a magnitude *as small as we please* by the continued division of a given magnitude.

To finish with Eudoxus we shall state his epochal definition of equal ratios which enabled mathematicians to treat irrational numbers as rigorously as the rationals. This was, essentially, the starting-point of one modern theory of irrationals.

"The first of four magnitudes is said to have the *same ratio* to the second that the third has to the fourth when, any whatever equimultiples [the same multiples] of the *first* and *third* being taken, and any other equimultiples of the *second* and *fourth,* the multiple of the *first* is greater than, equal to, or less than the multiple of the *second,* according as the multiple of the *third* is greater than, equal to, or less than the multiple of the *fourth.*"

Of the Greeks not yet named whose work influenced mathematics after the year 1600 only Apollonius need be mentioned here. Apollonius (260?–200? B.C.) carried geometry in the manner of Euclid—the way it is still taught to hapless beginners—far beyond the state in which Euclid (330?–275? B.C.) left it. As a geometer of this type—a *synthetic,* "pure" geometer—Apollonius is without a peer till Steiner in the nineteenth century.

If a cone standing on a circular base and extending indefinitely in both directions through its vertex is cut by a plane, the curve in which the plane intersects the surface of the cone is called a conic section. There are five possible kinds of conic sections: the ellipse; the hyperbola, consisting of two branches; the parabola, the path of a projectile in a vacuum; the circle; and a pair of intersecting straight lines. The ellipse, parabola and hyperbola are "mechanical curves" according to the Platonic formula; that is, these curves cannot be constructed by the use of straightedge and compass alone, although it is easy, with these implements, to construct any desired number of points lying on any one of these curves. The geometry of the conic sections, worked out to a high degree of perfection by Apollonius and his successors, proved to be of the highest importance in the celestial mechanics of the seventeenth and succeeding centuries. Indeed, had not the Greek geometers run ahead of Kepler it is unlikely that Newton could ever have come upon his law of universal gravitation, for which Kepler had



prepared the way with his laboriously ingenious calculations on the orbits of the planets.

Among the later Greeks and the Arabs of the Middle Ages Archimedes seems to have inspired the same awe and reverence that Gauss did among his contemporaries and followers in the nineteenth century, and that Newton did in the seventeenth and eighteenth. Archimedes was the undisputed chieftain of them all, "the old man," "the wise one," "the master," "the *great* geometer." To recall his dates, he lived in 287–212 B.C. Thanks to Plutarch more is known about his death than his life, and it is perhaps not unfair to suggest that the typical historical biographer Plutarch evidently thought the King of Mathematicians a less important personage historically than the Roman soldier Marcellus, into whose *Life* the account of Archimedes is slipped like a tissue-thin shaving of ham in a bull-choking sandwich. Yet Archimedes is today Marcellus' chief title to remembrance—and execration. In the death of Archimedes we shall see the first impact of a crassly practical civilization upon the greater thing which it destroyed—Rome, having half-demolished Carthage, swollen with victory and imperially purple with valor, falling upon Greece to shatter its fine fragility.

In body and mind Archimedes was an aristocrat. The son of the astronomer Pheidias, he was born at Syracuse, Sicily, and is said to have been related to Hieron II, tyrant (or king) of Syracuse. At any rate he was on intimate terms with Hieron and his son Gelon, both of whom had a high admiration for the king of mathematicians. His essentially aristocratic temperament expressed itself in his attitude to what would today be called applied science. Although he was one of the greatest mechanical geniuses of all time, if not the greatest when we consider how little he had to go on, the aristocratic Archimedes had a sincere contempt for his own practical inventions. From one point of view he was justified. Books could be written on what Archimedes did for applied mechanics; but great as this work was from our own mechanically biased point of view, it is completely overshadowed by his contributions to pure mathematics. We look first at the few known facts about him and the legend of his personality.

According to tradition Archimedes is a perfect museum specimen of the popular conception of what a great mathematician should be. Like Newton and Hamilton he left his meals untouched when he was deep in his mathematics. In the matter of inattention to dress he even surpasses Newton, for on making his famous discovery that a floating body loses in weight an amount equal to that of the liquid displaced, he leaped from the bath in which he had made the discovery by observing his own floating body, and dashed through the streets of Syracuse stark naked, shouting "*Eureka, eureka!*" (I have found it, I have found it!) What he had found was the first law of hydrostatics. According to the story a dishonest goldsmith had adulterated the gold of a crown for Hieron with silver and the tyrant, suspecting fraud, had asked Archimedes to put his mind on the problem. Any high school boy knows how it is solved by a simple experiment and some easy arithmetic on specific gravity; "the principle of Archimedes" and its numerous practical applications are meat for youngsters and naval engineers today, but the man who first saw through them had more than common insight. It is not definitely known whether the goldsmith was guilty; for the sake of the story it is usually assumed that he was.

Another exclamation of Archimedes which has come down through the centuries is "Give me a place to stand on and I will move the earth" (πᾶ βῶ καὶ κινῶ τὰν γᾶν, as he said it in Doric). He himself was strongly moved by his discovery of the laws of levers when he made his boast. The phrase would make a perfect motto for a modern scientific institute; it seems strange that it has not been appropriated. There is another version in better Greek but the meaning is the same.

In one of his eccentricities Archimedes resembled another great mathematician, Weierstrass. According to a sister of Weierstrass, he could not be trusted with a pencil when he was a young school teacher if there was a square foot of clear wallpaper or a clean cuff anywhere in sight. Archimedes beats this record. A sanded floor or dusted hard smooth earth was a common sort of "blackboard" in his day. Archimedes made his own occasions. Sitting before the fire he would rake out the ashes and draw in them. After stepping from the bath he would anoint himself with olive oil, according to the custom of the time, and then, instead of putting on his clothes, proceed to lose himself in the diagrams which he traced with a fingernail on his own oily skin.

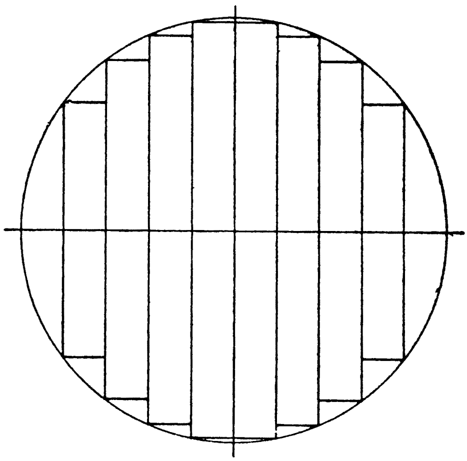
Archimedes was a lonely sort of eagle. As a young man he had studied for a short time at Alexandria, Egypt, where he made two life-long friends, Conon, a gifted mathematician for whom Archimedes had a high regard both personal and intellectual, and Eratosthenes, also a good mathematician but quite a fop. These two, particularly Conon, seem to have been the only men of his contemporaries with whom Archimedes felt he could share his thoughts and be assured of understanding. Some of his finest work was communicated by letters to Conon. Later, when Conon died, Archimedes corresponded with Dositheus, a pupil of Conon.

Leaving aside his great contributions to astronomy and mechanical invention we shall give a bare and inadequate summary of the principal additions which Archimedes made to pure and applied mathematics.

He invented general methods for finding the areas of curvilinear plane figures and volumes bounded by curved surfaces, and applied these methods to many special instances, including the circle, sphere, any segment of a parabola, the area enclosed between two radii and two successive whorls of a spiral, segments of spheres, and segments of surfaces generated by the revolution of rectangles (cylinders), triangles (cones), parabolas (paraboloids), hyperbolas (hyperboloids), and ellipses (spheroids) about their principal axes. He gave a method for calculating π (the ratio of the circumference of a circle to its diameter), and fixed π as lying between 3 1/7 and 3 10/71; he also gave methods for approximating to square roots which show that he anticipated the invention by the Hindus of what amount to periodic continued fractions. In arithmetic, far surpassing the incapacity of the unscientific Greek method of symbolizing numbers to write, or even to describe, large numbers, he invented a system of numeration capable of handling numbers as large as desired. In mechanics he laid down some of the fundamental postulates, discovered the laws of levers, and applied his mechanical principles (of levers) to calculate the areas and centers of gravity of several flat surfaces and solids of various shapes. He created the whole science of hydrostatics and applied it to find the positions of rest and of equilibrium of floating bodies of several kinds.

Archimedes composed not one masterpiece but many. How did he do it all? His severely economical, logical exposition gives no hint of the *method* by which he arrived at his wonderful results. But in 1906, J. L. Heiberg, the historian and scholar of Greek mathematics, made the dramatic discovery in Constantinople of a hitherto "lost" treatise of Archimedes addressed to his friend Eratosthenes: *On Mechanical Theorems, Method.* In it Archimedes explains how by weighing, in imagination, a figure or solid whose area or volume was unknown against a known one, he was led to the knowledge of the fact he sought; the fact being known it was then comparatively easy (for him) to prove it mathematically. In short he used his mechanics to advance his mathematics. This is one of his titles to a modern mind: *he used anything and everything that suggested itself as a weapon to attack his problems.*

To a modern all is fair in war, love, and mathematics; to many of the ancients, mathematics was a stultified game to be played according to the prim rules imposed by the philosophically-minded Plato. According to Plato only a straightedge and a pair of compasses were to be permitted as the implements of construction in geometry. No wonder the classical geometers hammered their heads for centuries against "the three problems of antiquity": to trisect an angle; to construct a cube having double the volume of a given cube; to construct a square equal to a circle. *None of these problems is possible with only straightedge and compass,* although it is hard to prove that the third is not, and the impossibility was finally proved only in 1882. All constructions effected with other implements were dubbed "mechanical" and, as such, for some mystical reason known only to Plato and his geometrizing God, were considered shockingly vulgar and were rigidly taboo in respectable geometry. Not till Descartes, 1985 years after the death of Plato, published his analytic geometry, did geometry escape from its Platonic straightjacket. Plato of course had been dead for sixty years or more before Archimedes was born, so he cannot be censured for not appreciating the lithe power and freedom of the methods of Archimedes. On the other hand, only praise is due Archimedes for not appreciating the old-maidishness of Plato's rigidly corseted conception of what the muse of geometry should be.



The second claim of Archimedes to modernity is also based upon his methods. Anticipating Newton and Leibniz by more than 2000 years he invented the integral calculus and in one of his problems anticipated their invention of the differential calculus. These two calculuses together constitute what is known as *the* calculus, which has been described as the most powerful instrument ever invented for the mathematical exploration of the physical universe. To take a simple example, suppose we wish to find the area of a circle. Among other ways of doing this we may slice the circle into any number of parallel strips of equal breadth, cut off the curved ends of the strips, so that the discarded bits shall total the least possible, by cuts perpendicular to the strips, and then add up the areas of all the resulting rectangles. This gives an approximation to the area sought. By increasing the number of strips indefinitely and taking the limit of the sum, we get the area of the circle. This (crudely described) process of taking the limit of the sum is called *integration*; the method of performing such summations is called the *integral calculus*. It was this calculus which Archimedes used in finding the area of a segment of a parabola and in other problems.

The problem in which he used the differential calculus was that of constructing a tangent at any given point of his spiral. If the angle which the tangent makes with any given line is known, the tangent can easily be drawn, for there is a simple construction for drawing a straight line through a given point parallel to a given straight line. The problem of finding the angle mentioned (for *any* curve, not merely for the spiral) is, in geometrical language, the main problem of the *differential* calculus. Archimedes solved this problem for his spiral. His spiral is the curve traced by a point moving with uniform speed along a straight line which revolves with uniform angular speed about a fixed point on the line. If anyone who has not studied the calculus imagines Archimedes' problem an easy one he may time himself doing it.

The life of Archimedes was as tranquil as a mathematician's should be if he is to accomplish all that is in him. All the action and tragedy of his life were crowded into its end. In 212 B.C. the second Punic war was roaring full blast. Rome and Carthage were going at one another hammer and tongs, and Syracuse, the city of Archimedes, lay temptingly near the path of the Roman fleet. Why not lay siege to it? They did.

Puffed up with conceit of himself ("relying on his own great fame," as Plutarch puts it), and trusting in the splendor of his "preparedness" rather than in brains, the Roman leader, Marcellus, anticipated a speedy conquest. The pride of his confident heart was a primitive piece of artillery on a lofty harp-shaped platform supported by eight galleys lashed together. Beholding all this fame and miscellaneous shipping descending upon them the timider citizens would have handed Marcellus the keys of the city. Not so Hieron. He too was prepared for war, and in a fashion that the practical Marcellus would never have dreamed of.

It seems that Archimedes, despising applied mathematics himself, had nevertheless yielded in peace time to the importunities of Hieron, and had demonstrated to the tyrant's satisfaction that mathematics can, on occasion, become devastatingly practical. To convince his friend that mathematics is capable of more than abstract deductions.

Archimedes had applied his laws of levers and pulleys to the manipulation of a fully loaded ship, which he himself launched single-handed. Remembering this feat when the war clouds began to gather ominously near, Hieron begged Archimedes to prepare a suitable welcome for Marcellus. Once more desisting from his researches to oblige his friend, Archimedes constituted himself a reception committee of one to trip the precipitate Romans. When they arrived his ingenious deviltries stood grimly waiting to greet them.

The harp-shaped turtle affair on the eight quinqueremes lasted no longer than the fame of the conceited Marcellus. A succession of stone shots, each weighing over a quarter of a ton, hurled from the super-catapults of Archimedes, demolished the unwieldy contraption. Crane-like beaks and iron claws reached over the walls for the approaching ships, seized them, spun them round, and sank or shattered them against the jutting cliffs. The land forces, mowed down by the Archimedean artillery, fared no better. Camouflaging his rout in the official bulletins as a withdrawal to a previously prepared position in the rear, Marcellus backed off to confer with his staff. Unable to rally his mutinous troops for an assault on the terrible walls, the famous Roman leader retired.

At last evincing some slight signs of military common sense, Marcellus issued no further "backs against the wall" orders of the day, abandoned all thoughts of a frontal attack, captured Megara in the rear, and finally sneaked up on Syracuse from behind. This time his luck was with him. The foolish Syracusans were in the middle of a bibulous religious celebration in honor of Artemis. War and religion have always made a bilious sort of cocktail; the celebrating Syracusans were very sick indeed. They woke up to find the massacre in full swing. Archimedes participated in the blood-letting.

His first intimation that the city had been taken by theft was the shadow of a Roman soldier falling across his diagram in the dust. According to one account the soldier had stepped on the diagram, angering Archimedes to exclaim sharply, "Don't disturb my circles!" Another states that Archimedes refused to obey the soldier's order that he accompany him to Marcellus until he had worked out his problem. In any event the soldier flew into a passion, unsheathed his glorious sword, and dispatched the unarmed veteran geometer of seventy five. Thus died Archimedes.

As Whitehead has observed, "No Roman lost his life because he was absorbed in the contemplation of a mathematical diagram."

CHAPTER THREE

Gentleman, Soldier, and Mathematician

DESCARTES

[Analytic geometry], far more than any of his metaphysical speculations, immortalized the name of Descartes, and constitutes the greatest single step ever made in the progress of the exact sciences.—JOHN STUART MILL

"I DESIRE ONLY TRANQUILLITY AND REPOSE." These are the words of the man who was to deflect mathematics into new channels and change the course of scientific history. Too often in his active life René Descartes was driven to find the tranquillity he sought in military camps and to seek the repose he craved for meditation in solitary retreat from curious and exacting friends. Desiring only tranquillity and repose, he was born on March 31, 1596 at La Haye, near Tours, France, into a Europe given over to war in the throes of religious and political reconstruction.

His times were not unlike our own. An old order was rapidly passing; the new was not yet established. The predatory barons, kings, and princelings of the Middle Ages had bred a swarm of rulers with the political ethics of highway robbers and, for the most part, the intellects of stable boys. What by common justice should have been thine was mine provided my arm was strong enough to take it away from thee. This may be an unflattering picture of that glorious period of European history known as the late Renaissance, but it accords fairly well with our own changing estimate, born of intimate experience, of what should be what in a civilized society.

On top of the wars for plunder in Descartes' day there was superimposed a rich deposit of religious bigotry and intolerance which incubated further wars and made the dispassionate pursuit of science a highly hazardous enterprise. To all this was added a comprehensive ignorance of the elementary rules of common cleanliness. From the point of view of sanitation the rich man's mansion was likely to be as filthy as the slums where the poor festered in dirt and ignorance, and the recurrent plagues which aided the epidemic wars in keeping the prolific population below the famine limit paid no attention to bank accounts. So much for the good old days.

On the immaterial, enduring side of the ledger the account is brighter. The age in which Descartes lived was indeed one of the great intellectual periods in the spotted history of civilization. To mention only a few of the outstanding men whose lives partly overlapped that of Descartes, we recall that Fermat and Pascal were his contemporaries in mathematics; Shakespeare died when Descartes was twenty; Descartes outlived Galileo by eight years, and Newton was eight when Descartes died; Descartes was twelve when Milton was born, and Harvey, the discoverer of the circulation of the blood, outlived Descartes by seven years, while Gilbert, who founded the science of electromagnetism, died when Descartes was seven.

René Descartes came from an old noble family. Although René's father was not wealthy his circumstances were a little better than easy, and his sons were destined for the careers of gentlemen—*noblesse oblige*—in the service of France. René was the third and last child of his father's first wife, Jeanne Brochard, who died a few days after René's birth. The father appears to have been a man of rare sense who did everything in his power to make up to his children for the loss of their mother. An excellent nurse took the mother's place, and the father, who married again, kept a constant, watchful, intelligent eye on his "young philosopher" who always wanted to know the cause of everything under the sun and the reason for whatever his nurse told him about heaven. Descartes was not exactly a precocious child, but his frail health forced him to expend what vitality he had in intellectual curiosity.

Owing to René's delicate health his father let lessons slide. The boy however went ahead on his own initiative and his father wisely let him do as he liked. When Descartes was eight his father decided that formal education could not be put off longer. After much intelligent inquiry he chose the Jesuit college at La Flèche as the ideal school for his son. The rector, Father Charlet, took an instant liking to the pale, confiding little boy and made a special study of his case. Seeing that he must build up the boy's body if he was to educate his mind, and noticing that Descartes seemed to require much more rest than normal boys of his age, the rector told him to lie in bed as late as he pleased in the mornings and not to leave his room till he felt like joining his companions in the classroom. Thereafter, all through his life except for one unfortunate episode near its close, Descartes spent his mornings in bed when he wished to think. Looking back in middle age on his schooldays at La Flèche, he averred that those long, quiet mornings of silent meditation were the real source of his philosophy and mathematics.

His work went well and he became a proficient classicist. In line with the educational tradition of the time much attention was put on Latin, Greek, and rhetoric. But this was only a part of what Descartes got. His teachers were men of the world themselves and it was their job to train the boys under their charge to be "gentlemen"—in the best sense of that degraded word—for their rôle in the world. When he left the school in August, 1612, in his seventeenth year, Descartes had made a life-long friend in Father Charlet and was almost ready to hold his own in society. Charlet was only one of the many friends Descartes made at La Flèche; another, Mersenne (later Father), the famous amateur of science and mathematics, had been his older chum and was to become his scientific agent and protector-in-chief from bores.

Descartes' distinctive talent had made itself evident long before he left school. As early as the age of fourteen, lying meditating in bed, he had begun to suspect that the "humanities" he was mastering were comparatively barren of human significance and certainly not the sort of learning to enable human beings to control their environment and direct their own destiny. The authoritative dogmas of philosophy, ethics, and morals offered for his blind acceptance began to take on the aspect of baseless superstitions. Persisting in his childhood habit of accepting nothing on mere authority, Descartes began unostentatiously questioning the alleged demonstrations and the casuistical logic by which the good Jesuits sought to gain the assent of his reasoning faculties. From this he rapidly passed to the fundamental doubt which was to inspire his life-work: how do we *know* anything? And further, perhaps more importantly, if we cannot say definitely that we know anything, how are we ever to find out those things which we may be capable of knowing?

On leaving school Descartes thought longer, harder, and more desperately than ever. As a first fruit of his meditations he apprehended the heretical truth that logic of itself—the great method of the schoolmen of the Middle Ages which still hung on tenaciously in humanistic education—is as barren as a mule for any creative human purpose. His second conclusion was closely allied to his first: compared to the demonstrations of mathematics—to which he took like a bird to the air as soon as he found his wings—those of philosophy, ethics, and morals are tawdry shams and frauds. How then, he asked, shall we ever find out anything? By the scientific method, although Descartes did not call it that: by *controlled experiment* and the application of rigid mathematical reasoning to the results of such experiment.

It may be asked what he got out of his rational skepticism. One fact, and only one: "I exist." As he put it, "*Cogito ergo sum*" (I think, therefore I am).

By the age of eighteen Descartes was thoroughly disgusted with the aridity of the studies on which he had put so much hard labor. He resolved to see the world and learn something of life as it is lived in flesh and blood and not in paper and printers' ink. Thanking God that he was well enough off to do as he pleased he proceeded to do it. By an understandable overcorrection of his physically inhibited childhood and youth he now fell upon the pleasures appropriate to normal young men of his age and station and despoiled them with both hands. With several other young blades hungering for life in the raw he quit the depressing sobriety of the paternal estate and settled in Paris. Gambling being one of the accomplishments of a gentleman in that day, Descartes gambled with enthusiasm—and some success. Whatever he undertook he did with his whole soul.

This phase did not last long. Tiring of his bawdy companions, Descartes gave them the slip and took up his quarters in plain, comfortable lodgings in what is now the suburb of Saint-Germain where, for two years, he buried himself in incessant mathematical investigation. His gay deeds at last found him out, however, and his hare-brained friends descended whooping upon him. The studious young man looked up, recognized his friends, and saw that they were one and all intolerable bores. To get a little peace Descartes decided to go to war.

Thus began his first spell of soldiering. He went first to Breda, Holland, to learn his trade under the brilliant Prince Maurice of Orange. Being disappointed in his hopes for action under the Prince's colors, Descartes turned a disgusted back on the peaceful life of the camp, which threatened to become as exacting as the hurly-burly of Paris, and hastened to Germany. At this point of his career he first showed symptoms of an amiable weakness which he never outgrew. Like a small boy trailing a circus from village to village Descartes seized every favorable opportunity to view a gaudy spectacle. One was now about to come off at Frankfurt, where Ferdinand II was to be crowned. Descartes arrived in time to take in the whole rococo show. Considerably cheered up he again sought his profession and enlisted under the Elector of Bavaria, then waging war against Bohemia.

The army was lying inactive in its winter quarters near the little village of Neuburg on the banks of the Danube. There Descartes found in plenty what he had been seeking, tranquillity and repose. He was left to himself and he found himself.

The story of Descartes' "conversion"—if it may be called that—is extremely curious. On St. Martin's Eve, November 10, 1619, Descartes experienced three vivid dreams which, he says, changed the whole current of his life. His biographer (Baillet) records the fact that there had been considerable drinking in celebration of the saint's feast and suggests that Descartes had not fully recovered from the fumes of the wine when he retired. Descartes himself attributes his dreams to quite another source and states emphatically that he had touched no wine for three months before his elevating experience. There is no reason to doubt his word. The dreams are singularly coherent and quite unlike those (according to experts) inspired by a debauch, especially of stomach-filling wine. On the surface they are easily explicable as the subconscious resolution of a conflict between the dreamer's desire to lead an intellectual life and his realization of the futility of the life he was actually living. No doubt the Freudians have analyzed these dreams, but it seems unlikely that any analysis in the classical Viennese manner could throw further light on the invention of analytic geometry, in which we are chiefly interested here. Nor do the several mystic or religious interpretations seem likely to be of much assistance in this respect.

In the first dream Descartes was blown by evil winds from the security of his church or college toward a third party which the wind was powerless to shake or budge; in the second he found himself observing a terrific storm with the unsuperstitious eyes of science, and he noted that the storm, once seen for what it was, could do him no harm; in the third he dreamed that he was reciting the poem of Ausonius which begins, "*Quod vitae secatabor iter?*" (What way of life shall I follow?)

There was much more. Out of it all Descartes says he was filled with "enthusiasm" (probably intended in a mystic sense) and that there had been revealed to him, as in the second dream, the magic key which would unlock the treasure house of nature and put him in possession of the true foundation, at least, of all the sciences.

What was this marvelous key? Descartes himself does not seem to have told anyone explicitly, but it is usually believed to have been nothing less than the application of algebra to geometry, analytic geometry in short and, more generally, the exploration of natural phenomena by mathematics, of which mathematical physics today is the most highly developed example.

November 10, 1619, then, is the official birthday of analytic geometry and therefore also of modern mathematics. Eighteen years were to pass before the method was published. In the meantime Descartes went on with his soldiering. On his behalf mathematics may thank Mars that no half-spent shot knocked his head off at the battle of Prague. A score or so of promising young mathematicians a few years short of three centuries later were less lucky, owing to the advance of that science which Descartes' dream inspired.

As never before the young soldier of twenty two now realized that if he was ever to find truth he must first reject absolutely all ideas acquired from others and rely upon the patient questioning of his own mortal mind to show him the way. All the knowledge he had received from authority must be cast aside; the whole fabric of his inherited moral and intellectual ideas must be destroyed, to be refashioned more enduringly by the primitive, earthy strength of human reason alone. To placate his conscience he prayed the Holy Virgin to help him in his heretical project. Anticipating her assistance he vowed a pilgrimage to the shrine of Notre-Dame de Lorette and proceeded forthwith to subject the accepted truths of religion to a scorching, devastating criticism. However, he duly discharged his part of the contract when he found the opportunity.

In the meantime he continued his soldiering, and in the spring of 1620 enjoyed some very real fighting at the battle of Prague. With the rest of the victors Descartes entered the city chanting praises to

God. Among the terrified refugees was the four-year-old Princess Elisabeth,* who was later to become Descartes' favorite disciple.

At last, in the spring of 1621, Descartes got his bellyful of war. With several other gay gentlemen soldiers he had accompanied the Austrians into Transylvania, seeking glory and finding it—on the other side. But if he was through with war for the moment he was not yet ripe for philosophy. The plague in Paris and the war against the Huguenots made France even less attractive than Austria. Northern Europe was both peaceful and clean; Descartes decided to pay it a visit. Things went well enough till Descartes dismissed all but one of his bodyguard before taking boat for east Frisia. Here was a Heaven-sent opportunity for the cut-throat crew. They decided to knock their prosperous passenger on the head, loot him, and pitch his carcase to the fish. Unfortunately for their plans Descartes understood their language. Whipping out his sword he compelled them to row him back to the shore, and once again analytic geometry escaped the accidents of battle, murder, and sudden death.

The following year passed quietly enough in visits to Holland and Rennes, where Descartes' father lived. At the end of the year he returned to Paris, where his reserved manner and somewhat mysterious appearance immediately got him accused of being a Rosicrucian. Ignoring the gossip, Descartes philosophized and played politics to get himself a commission in the army. He was not really disappointed when he failed, as he was left free to visit Rome where he enjoyed the most gorgeous spectacle he had yet witnessed, the ceremony celebrated every quarter of a century by the Catholic Church. This Italian interlude is of importance in Descartes' intellectual development for two reasons. His philosophy, so far as it fails to touch the common man, was permanently biased against that lowly individual by the fill which the bewildered philosopher got of unwashed humanity gathered from all corners of Europe to receive the papal benediction. Equally important was Descartes' failure to meet Galileo. Had the mathematician been philosopher enough to sit for a week or two at the feet of the father of modern science, his own speculations on the physical universe might have been less fantastic. All that Descartes got out of his Italian journey was a grudging jealousy of his incomparable contemporary.

* Daughter of Frederick, Elector Palatine of the Rhine, and King of Bohemia, and a granddaughter of James I of England.

Immediately after his holiday in Rome, Descartes enjoyed another bloody spree of soldiering with the Duke of Savoy, in which he so distinguished himself that he was offered a lieutenant generalship. He had sense enough to decline. Returning to the Paris of Cardinal Richelieu and the swashing D'Artagnan—the latter near-fiction, the former less credible than a melodrama—Descartes settled down to three years of meditation. In spite of his lofty thoughts he was no graybearded savant in a dirty smock, but a dapper, well dressed man of the world, clad in fashionable taffeta and sporting a sword as befitted his gentlemanly rank. To put the finishing touch to his elegance he crowned himself with a sweeping, broad-brimmed, ostrich-plumed hat. Thus equipped he was ready for the cut-throats infesting church, state, and street. Once when a drunken lout insulted Descartes' lady of the evening, the irate philosopher went after the rash fool quite in the stump-stirring fashion of D'Artagnan, and having flicked the sot's sword out of his hand, spared his life, not because he was a rotten swordsman, but because he was too filthy to be butchered before a beautiful lady.

Having mentioned one of Descartes' lady friends we may dispose of all but two of the rest here. Descartes liked women well enough to have a daughter by one. The child's early death affected him deeply. Possibly his reason for never marrying may have been, as he informed one expectant lady, that he preferred truth to beauty; but it seems more probable that he was too shrewd to mortgage his tranquillity and repose to some fat, rich, Dutch widow. Descartes was only moderately well off, but he knew when he had enough. For this he has been called cold and selfish. It seems juster to say that he knew where he was going and that he realized the importance of his goal. Temperate and abstemious in his habits he was not mean, never inflicting on his household the Spartan regimen he occasionally prescribed for himself. His servants adored him, and he interested himself in their welfare long after they had left his service. The boy who was with him at his death was inconsolable for days at the loss of his master. All this does not sound like selfishness.

Descartes also has been accused of atheism. Nothing could be farther from the truth. His religious beliefs were unaffectedly simple in spite of his rational skepticism. He compared his religion, indeed, to the nurse from whom he had received it, and declared that he found it as comforting to lean upon one as on the other. A rational mind is sometimes the queerest mixture of rationality and irrationality on earth.

Another trait affected all Descartes' actions till he gradually outgrew it under the rugged discipline of soldiering. The necessary coddling of his delicate childhood infected him with a deep tinge of hypochondria, and for years he was chilled by an oppressive dread of death. This, no doubt, is the origin of his biological researches. By middle age he could say sincerely that nature is the best physician and that the secret of keeping well is to lose the fear of death. He no longer fretted to discover means of prolonging existence.

His three years of peaceful meditation in Paris were the happiest of Descartes' life. Galileo's brilliant discoveries with his crudely constructed telescope had set half the natural philosophers of Europe to pottering with lenses. Descartes amused himself in this way, but did nothing of striking novelty. His genius was essentially mathematical and abstract. One discovery which he made at this time, that of the principle of virtual velocities in mechanics, is still of scientific importance. This really was first-rate work. Finding that few understood or appreciated it, he abandoned abstract matters and turned to what he considered the highest of all studies, that of man. But, as he dryly remarks, he soon discovered that the number of those who understand man is negligible in comparison with the number of those who think they understand geometry.

Up till now Descartes had published nothing. His rapidly mounting reputation again attracted a horde of fashionable dilettantes, and once more Descartes sought tranquillity and repose on the battlefield, this time with the King of France at the siege of La Rochelle. There he met that engaging old rascal Cardinal Richelieu, who was later to do him a good turn, and was impressed, not by the Cardinal's wiliness, but by his holiness. On the victorious conclusion of the war Descartes returned with a whole skin to Paris, this time to suffer his second conversion and abandon futilities forever.

He was now (1628) thirty two, and only his miraculous luck had preserved his body from destruction and his mind from oblivion. A stray bullet at La Rochelle might easily have deprived Descartes of all claim to remembrance, and he realized at last that if he was ever to arrive it was high time that he be on his way. He was aroused from his sterile state of passive indifference by two Cardinals, De Bérulle and De Bagné, to the first of whom in particular the scientific world owes an everlasting debt of gratitude for having induced Descartes to publish.

The Catholic clergy of the time cultivated and passionately loved the sciences, in grateful contrast to the fanatical Protestants whose bigotry had extinguished the sciences in Germany. On becoming acquainted with De Bérulle and De Bagné, Descartes blossomed out like a rose under their genial encouragement. In particular, during soirées at De Bagné's, Descartes spoke freely of his new philosophy to a M. de Chandoux (who was later hanged for counterfeiting, not a result of Descartes' lessons in casuistry, let us hope). To illustrate the difficulty of distinguishing the true from the false Descartes undertook to produce twelve irrefutable arguments showing the falsity of any incontestable truth and, conversely, to do the like for the truth of any admitted falsehood. How then, the bewildered listeners asked, shall mere human beings distinguish truth from falsehood? Descartes confided that he had (what he considered) an infallible method, drawn from mathematics, for making the required distinction. He hoped and planned, he said, to show how his method could be applied to science and human welfare through the medium of mechanical invention.

De Bérulle was profoundly stirred by the vision of all the kingdoms of the earth with which Descartes had tempted him from the pinnacle of philosophic speculation. In no uncertain terms he told Descartes that it was his duty to God to share his discoveries with the world, and threatened him with hell-fire—or at least the loss of his chance of heaven—if he did not. Being a devout practising Catholic Descartes could not possibly resist such an appeal. He decided to publish. This was his second conversion, at the age of thirty two. He straightway retired to Holland, where the colder climate suited him, to bring his decision to realization.

For the next twenty years he wandered about all over Holland, never settling for long in any one place, a silent recluse in obscure villages, country hotels and out-of-the-way corners of great cities, methodically carrying on a voluminous scientific and philosophical correspondence with the leading intellects of Europe, using as intermediary the trusted friend of his school days at La Flèche, Father Mersenne, who alone knew the secret at any time of Descartes' address. The parlor of the cloister of the Minims, not far from Paris, became the