GENERAL PHYSICS

AN ELEMENTARY TEXT-BOOK FOR COLLEGES

HENRY CREW, Ph. D.

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IN these days of numerous text-books the author who dares still further to increase the number owes to the public, at least, an explanation in which he shall set forth his purpose, however far short of accomplishment he may have fallen.

Accordingly, in this volume, the author's hopes have been:—

(i) To prepare a text-book adapted to the needs of first-year students in physics at Northwestern University. One fear in this connection is that the transfer of a course of lectures from flexible manuscript to rigid type metal may correspond to that stage of development which, in the life history of an animal, is known as *rigor mortis*. This difficulty, it is hoped, may be avoided by varying, from year to year, the illustrative phenomena employed in the lecture demonstrations.

(ii) To keep the treatment elementary, and yet include all the fundamental principles of physics; and at the same time to bind them together with "connective tissue" in such a way as to make clear to the student the essential unity of the subject.

(iii) Not merely, or even mainly, to impart information, but to set before the student a large and compact body of truth obtained by a method which shall remain for him, throughout life, a pattern and norm of clear and correct thinking.

To succeed completely in an undertaking so ambitious as that suggested by the threefold purpose just stated is more than the author dares to hope.

To my friend, Mr. John Mackenzie of Minneapolis, I am indebted for those paragraphs in Chapter III which describe Professor Osborne Reynolds' remarkable theory of gravitation.

The imperfections of the present text would have been many more except for the clever revision of the proof by my colleague, Professor R. R. Tatnall, and by my friend, Professor A. A. Knowlton of the Armour Institute. Special thanks are due Mr. J. H. Spencer of the United States Weather Bureau,

to the *Youth's Companion*, and to the De Laval Steam Turbine Company, for the use of cuts. Mr. John A. Fleming of the Carnegie Institution was good enough to revise those paragraphs dealing with the secular variation of the earth's magnetic field. To my sister, Caroline L. Crew, I am indebted for the excision of many awkward English constructions.

H. C.

EVANSTON, ILLINOIS,
April 8, 1908.

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GENERAL PHYSICS

INTRODUCTORY

PHYSICS, in the present sense of the word, is not an ancient science, but a creation of the last three hundred years. Compared with astronomy and medicine, it is a mere upstart. When, however, one considers the work of the Egyptians in handling tremendous blocks of stone, such as those used in the construction of the pyramids, or the work of the Greeks in devising powerful catapults, or the work of the Romans in distributing water through well-constructed aqueducts, or any one of their numerous clever devices, such as the steelyards, locks, and surgical instruments which have been recovered at Pompeii, it becomes evident that the ancients were familiar with a wide range of physical science. That there were among them keen observers is patent to every one who has considered the Hermes of Praxiteles or the columns of the Parthenon. The skill of the ancients in hunting and the accuracy with which they observed the motions of the planets are proverbial.

Nor were they wanting in the desire for a unitary view of the world about them. This search for unity is evinced in the idea of Heraclitus that igneous vapor (fire) was the one great principle, in the idea of Thales that aqueous vapor was the key to the situation, in the idea of Anaximenes that air was the essential principle, and in the notion of Democritus that the origin of all things was to be sought in the atom. The failure of these various views gave rise to the general opinion that four "elements"— earth, air, fire, and water — constituted an irreducible minimum for the physical universe. The Ptolemaic system of crystalline spheres, cycles, and epicycles is another product of their eagerness for unity. Well, then, in what respect does modern physical science differ from the ancient? Why is the beginning of this science supposed to lie only some three hundred years back of the present?

The answer to this question, like that to most others which begin with "why," is not easy; and all that can be done here is to point out briefly *how* ancient differs from modern science rather than to explain *why* this difference exists. During the fifteenth century there seemed to come over the minds of thinking men a change, marked especially by an increased freedom from the trammels of authority. Such freedom had already, even as early as the fourteenth century, made itself evident in painting, architecture, and music. A new literature had found expression in Dante, Petrarch, and Boccaccio, whose works were now being disseminated by the newly invented art of printing. An increased boldness was shown by navigators, which culminated in the discovery of America. This same century was marked by the birth of Copernicus, with Paracelsus and Leonardo da Vinci for contemporaries. Without shedding any new light upon the "cause" of this changed attitude of mind, it may perhaps be best described as "the spirit of the Renaissance." Previous to this time, those great generalizations which we now call "physical laws" seem to have been rather fortuitous and unfruitful; after this period, the search for law, for the constant elements in the behavior of nature, seems to have been more a matter of deliberate and definite plan, while the results of the search have been tremendously prolific of other important results.

It was upon such a background as this, and into such a changed world as this, that on the 15th of February, 1564, at Pisa, in Italy, the founder of modern physics was born. But, while all science is the creation of man, it is equally true that no science is the creation of any single man. In what sense, therefore, can Galileo be spoken of as the founder of modern physics? Just this: he it was who first discovered the fundamental principles of modern dynamics, the one branch of learning upon which the entire superstructure of the physical sciences has been erected. His formulation of these principles was, as we shall later see, imperfect; but his suggestion that bodies change their motion when, and only when, acted upon by some external force — or, if you please, the substitution of the conception "force" in place of "cause" of motion — has proven to be the most fertile principle, as well as the first fundamental and perfectly general principle, of modern dynamics, a principle

which finds its later and more perfect expression in Newton's Three Laws of Motion. Among the ancients, Archimedes (287–212 B.C.) appears to be the one man who approached the modern method in physics. He established a firm foundation for the science of statics — the conditions of equilibrium for bodies at rest; but this is a very special case of dynamics, and the achievements of Archimedes, which early attracted the attention of Galileo, — remarkable and valuable as they impress us to-day, — bore little fruit until the idea of force as the condition for change of motion was introduced by the great Italian. From Galileo's time, on, motions cease to be classified as "natural" and "violent." *All* motions are henceforth "natural," and are proper subjects for special investigation. The faith and skill in experiment exhibited by Galileo set a new pace, one which had possibly never before been reached, and one which has certainly seldom since been surpassed.

The experimental method once established, new facts and new generalizations accumulated with marvelous rapidity; and the subject of natural philosophy, enriched by the labors of Hooke, Boyle, and Newton in England, Descartes and Mersenne in France, Huygens in Holland, Torricelli and other pupils of Galileo in Italy, became so bulky that certain natural cleavage lines began to appear.

Certain phenomena were soon recognized as peculiar to living bodies — bodies which possess the ability to assimilate food, the power of reproduction, sensation, etc., and these came to be studied under the head of natural history and medicine. This entire group, including, as it now does, botany, zoölogy, physiology, anatomy, paleontology (as distinguished from geology), bacteriology, etc., has, by wide consent, been given the name of the **natural sciences**. All matter, other than that of living bodies, is said to be inanimate; and it is with this class of bodies especially that the **physical sciences** have to do.

In the early days, nearly all the phenomena of the physical sciences were studied under the head of **Natural Philosophy**, a term which was employed in contradistinction to **Natural History**, and one which included such diverse subjects as the motions of the solar system, engineering structures, and meteorology. But Physics, or Natural Philosophy, in its narrow and proper sense, does not deal with *all* the phenomena of dead

matter. There are, for instance, certain bodies which are so distant from us that they can be studied only by specially designed instruments such as the telescope. Accordingly, the study of the fixed stars, the planets, comets, etc., is pursued under a special department of physical science called **Astronomy.** There is another exceedingly large and important branch of physical science which deals especially with those changes which occur in homogeneous bodies — mainly changes of composition. The systematic study of these changes is carried on under the head of **Chemistry.**

Those properties of inanimate bodies which are not primarily connected with a change in their chemical composition are then the especial subjects of study pursued under the head of **Physics.**

Besides these there are other important, but more specialized, branches of physical science which deal with particular groups of bodies. Thus the comprehensive science which takes up the study of the earth's crust from a physical and chemical standpoint is called **Geology**; while **Meteorology**, a science of great significance and of increasing practical value, deals exclusively with the physical phenomena of the earth's atmosphere.

Built upon the physical and natural sciences, is still another group of studies distinguished from the preceding "pure" sciences mainly by the purpose and attitude of mind on the part of the student. Such are the various branches of engineering, which are largely branches of applied physics; such also is the study of modern medicine, a science which has become immediately dependent upon Biology, Chemistry, Physics, Physiology, and Anatomy.

In attempting to roughly locate Physics among the various sciences, we have so far considered only those groups of phenomena which are associated with matter, and have left to one side the phenomena of mind, also the historical sciences which deal with the purposes and the controlling wills of individuals and nations. To one side also have been left those sciences which deal with the consensus of sane opinion and with the expression of the will of the entire race, sciences which Münsterberg calls "Normative" and which include Logic and Mathematics.

The place of Physics in the family of the sciences is then a very modest one, for it deals only with a particular group of phenomena, associated with inanimate matter, and limited essen-

tially to those changes in which the composition of the various substances involved remains unaltered. Within this limited domain the purpose of the physicist is to discover and to describe the universal and constant modes in the behavior of nature. Very often these results can be given a simple and accurate mathematical formulation. With what splendid success this work has been pushed forward will appear when we come to study in more detail the work of the men who have followed the great Galileo, men such as Newton, Faraday, and Maxwell in England; Fresnel and Ampère in France; Gauss, Weber, Kirchhoff, Helmholtz, and Hertz in Germany; Volta and Melloni in Italy; Franklin, Henry, and Rowland in America.

CHAPTER I

KINEMATICS

Position

1. Kirchhoff (1824–1886), a distinguished German physicist, first clearly pointed out that the aim of the physicist is to describe the motions of bodies completely and in the simplest possible manner. But since a body is a limited portion of matter, which we may consider as made up of a number of small particles, it will be simpler to consider first the motion of a *particle* and then pass to the consideration of *bodies* in motion. However, since motion is merely change of position, it will be convenient first to learn how the position of a particle is defined. The word "particle" is used always to denote a body so small that its dimensions may be neglected in comparison with other distances involved. Thus a baseball may be considered a particle when one is thinking only of its path through the air; but in the pitcher's hands it becomes a body of very appreciable size, a body which he sets into rotation about a definite axis in order to give it any desired "curve."

Position of a Particle in a Plane

2. When geographers describe the position of a town, they assume that the reader already knows the position of some point on the earth, which may be called a point of reference. Such a point of reference for the surface of the earth is the intersection of the meridian through Greenwich with the equator. This fixed, the location of a town is definitely described when two things are known about it; namely, its *latitude* and its *longitude*. Thus the island of St. Helena is in 15° 55′ south latitude and 5° 43′ west longitude.

A similar method is frequently used in physics; *e.g.*, the position of a point P (Fig. 1), with reference to another point O, *in the same plane*, is described when two mutually rectangu-

lar straight lines are drawn through the point O, and the perpendicular distances of the point P from these two lines respectively are given. **The point of reference O is generally called the origin. The horizontal line of reference $\overline{OX}$** (Fig. 1) **is generally called the axis of X; and the vertical line of reference $\overline{OY}$ the axis of Y. The longitude x of the point P is called its abscissa; while the latitude y is known as its ordinate.** Abscissas are positive when drawn to the right from the axis of Y; negative when drawn to the left. Ordinates are positive when drawn upward from the axis of X; negative when drawn downward. These quantities x and y are known as the **rectangular coördinates** of the point P; they are a special kind of Cartesian coördinates, so called after Descartes, who first employed an algebra of this kind.

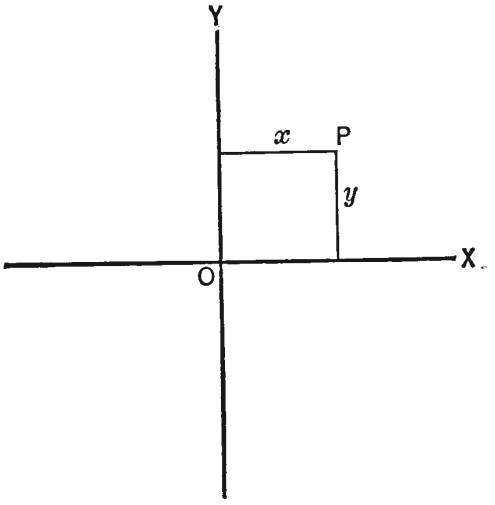


FIG. 1. — Showing one method of locating a point in a plane.

In the laboratory we have frequent occasions to draw curves, and this is generally done by locating points on coördinate paper in the manner just described.

Navigators and surveyors generally employ a different method of locating one point with reference to another in the same plane. **They give the straight-line distance from the first point to the second, then they give the direction of the line joining the points.** This method also is frequently used in physics and is known as the method of polar coördinates. The idea will be clear from Fig. 2, where O is the point of reference and $\overline{OX}$ any known direction. There is no doubt about the exact position of a particle at P as soon as we know the distance r between the points O and P, and the angle θ between the lines OP and OX. **The angular distance θ we**

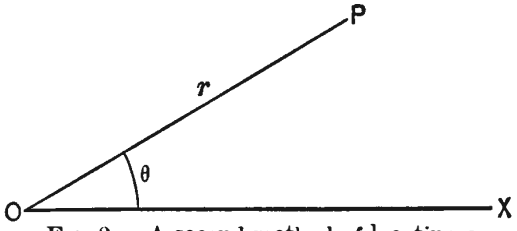


FIG. 2. — A second method of locating a point in a plane.

shall always consider positive when measured in a direction contrary to that in which the hands of a clock move, a direction which is called counter-clockwise. In like manner, θ is considered negative when measured in a clockwise sense. The line $\overline{OX}$ we may call the line of reference.

Position of a Particle in Space

3. If a point on the earth's surface is to be described still more completely, we may choose the sea level as a plane of reference, and give the altitude of the point. So in mathematics and physics one frequently describes a point by locating it first in a certain plane as described in the preceding sections, and then stating the position of this plane with respect to some other plane of reference. It will be observed that, in general, *three specifications are needed to completely describe the position of one point with reference to another;* and when the numerical values of these three coördinates are once known, the position of the point is given without ambiguity. Thus if the rectangular coördinates of a particle in space have the values, —

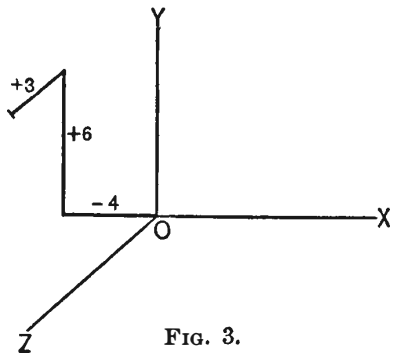


Fig. 3.

$$\begin{cases} x = -4, \\ y = +6, \\ z = +3, \end{cases}$$

one can proceed to locate it as indicated in Fig. 3.

Position Vectors

4. But a third and simpler way to think of the position of a point is to consider it as completely specified by a single straight line running from the point of reference to the point under consideration. If in Fig. 4 we know, not only the length, but also the direction and sense of the straight line $\overline{OP}$, we know all that can possibly be known about the position of the particle at P.

To any quantity such as $\overline{OP}$, possessing direction, sense, and amount, the name "vector" has been given. This class of quantities includes many of the most important studied in physics.

A vector quantity is simply one which *resembles a limited straight line* in three particulars; namely, *in the possession of direction, sense, and numerical magnitude.* A vector which locates one point with reference to another, as in Fig. 4, is called a position vector, and is a type of vector quantities in general, of which we shall meet many examples in later pages. The usefulness of the vector, in representing the position of a point, lies in its clearness and simplicity; for purposes of numerical calculation it has small value.

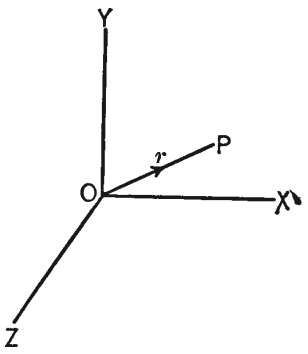


FIG. 4.

DIGRESSION ON SCALAR QUANTITIES

5. The mere length of a straight line is the same in whatever direction the line may lie. The *length* of a straight line is, therefore, *not* a vector quantity. It is independent of direction. **Quantities which do not have direction are called scalars.**

Philadelphia is a city 90 miles from New York. This distance is a scalar quantity. The information contained in this statement tells me only that Philadelphia lies somewhere on a circle of 90 miles radius described with New York as a center. But "90 miles southwest of New York" gives me a vector quantity, which locates the position of Philadelphia without ambiguity. A pound of sugar, three yards of calico, two gallons of water, a volume of anything, any amount of matter, are illustrations of scalars.

SUMMARY

6. Summarizing, then, the position of a particle is nearly always described by one or the other of the three following methods:—

(i) *Rectangular coördinates*, in which the position of a point is defined by giving its distance from each of three mutually perpendicular planes. These distances are called the rectangular coördinates of the particle and are usually denoted by x, y, z. The three mutually perpendicular reference planes are indicated in Fig. 3 by *YOZ*, *ZOX*, and *XOY*.

(ii) *Polar coördinates*, in which the position of a particle is described by its distance r from the origin, and by its latitude and longitude on a sphere whose center is at the origin and whose radius is the distance r just described. These three quantities are called the polar coördinates of the particle and are usually denoted by the letters r, θ, and ϕ respectively.

(iii) *Vectors*, a graphical method in which the position of a particle is located as the terminal point of a single straight line drawn in a definite sense and in a definite direction from the origin.

Position of a Body

7. Having thus far considered the position of a *particle* only, we now take up the position of a *body* which has already been defined as a limited portion of matter. For the present let us deal only with perfectly rigid solids, that is, with bodies whose size and shape are each constant. If we consider a body merely as a system of rigidly connected particles, it is evident that its position is completely defined only when the position of *each* particle in the system is known. But since the body is rigid, this is equivalent to saying that its position is completely determined when we know the positions of any three points in the body, provided of course that these points do not lie in the same straight line. In other words, if a straight line and one point outside this line be located in a body, the position of the entire body is completely defined. It is clear, therefore, that a body may have many different positions in which two points in the body are fixed; for under this condition it may be rotated through any angle about the straight line joining these two points. As we shall see later, it is often quite as necessary to consider the angular as well as the linear position of a body; while for a single particle angular position is a term which has no meaning.

Measurement of Lengths

8. The actual determination of the position of any point in the world about us leads at once to the measurement of a distance, by which is meant simply the comparison of any assigned distance with a given distance which we agree to take as unity.

As later experience will show, comparison is the essential feature of all measurement. Ten yards of cloth is simply a piece of cloth which is ten times as long as a certain piece of wood called a yard stick.

Units of Length. Centimeter. Foot

9. Almost the entire scientific world has agreed to take as the standard length the distance between two marks on a certain platinum-iridium bar preserved in the International Metric Bureau at Sèvres, near Paris. This distance is called a meter. The one hundredth part of the distance between these two marks is taken as the unit and is called a centimeter. For many laboratory measurements this smaller unit will be found more convenient than the meter.

It will be observed that this definition is purely arbitrary, a mere matter of convention. The fact that the meter is so chosen as to be nearly equal to one ten-millionth of the distance from the earth's equator to its north pole is interesting; but it does not enter into the definition of the unit of length. As a matter of fact, the length of the earth's quadrant has been found to be more nearly 10,000,880 meters. The student who is not familiar with the metric system will find it interesting to estimate his own height in centimeters, also to guess at the length and width of this page in centimeters; and then test the accuracy of his estimates by actual measurement with a meter stick.

English and American engineers generally employ the yard as the standard of length and the foot as the unit of length.

In Great Britain the yard is, by definition, the distance between two rulings on a certain bronze bar preserved in the Standards Office at London. In the United States the yard is defined as 36/39.37 of the standard meter at Sèvres.

Measurement of Areas. Unit of Area

10. Having settled upon the centimeter as a unit of length, the simplest possible unit of area is the area of one square centimeter. In consideration of this fact, the unit of area almost universally employed in science is the square centimeter. Very seldom, indeed, are areas measured by direct com-

parison with a unit of area. More frequently it will be found convenient to measure the *linear* dimensions of the surface in question, and, from these, to *compute* by geometry the area of the surface. Engineers, of course, employ the square foot as the unit of area.

Measurement of Volumes. Unit of Volume

11. Standards of volume, such as the bushel, the quart, the cubic foot, are already familiar. **In scientific work the cubic centimeter is adopted as the unit of** volume, and for the same reasons that led to the adoption of the square centimeter as the unit of area. An acquaintance with the metric system makes the wisdom of this choice at once apparent.

In the case of volumes whose linear dimensions can be measured, *e.g.* a parallelopipedon or a sphere, it is frequently more convenient to compute the volume than to measure it directly.

Angular Position. Unit Angle

12. When two straight lines do not coincide in direction, they are said to have different angular positions. The difference of their directions is the angular displacement of one line with respect to the other. This angular displacement is expressed by surveyors and astronomers in degrees. But navigators use "points" of the compass, or $11\frac{1}{4}°$, as units. In physics and mathematics the unit of angle mostly employed in the description and discussion of phenomena is different from either of these. It is defined as follows: If two straight lines lying in the same plane differ in direction, they will intersect. About the point of intersection as center, and in the plane of these two lines, draw a circle. **These lines are said to include a unit angle when they intercept an arc of this circle equal in length to its radius. See Fig. 5. This unit angle is called the radian.**

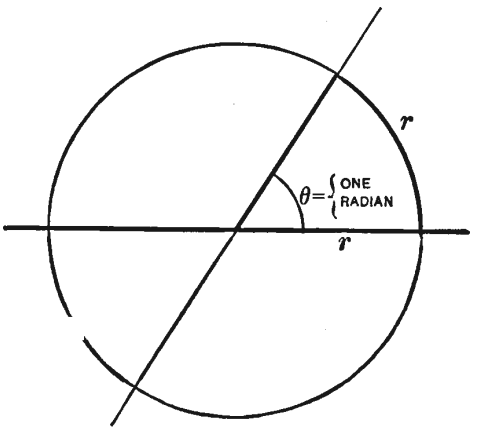


Fig. 5. — Unit angle.

It is evident, since the circumference of a circle is a length equal to 2π times the radius, that in one right angle there will be one quarter of 2π radians, *i.e.* $\frac{\pi}{2}$ radians. Compute, and on the margin of this page record the number of degrees in one radian.

Problems

1. Explain just how it is that any limited straight line is a vector quantity, while the length of such a line is a scalar quantity.

2. What light does the etymology of the word "vector" shed upon its present meaning?

3. How many radians in the angle subtended by a semicircle?

4. The volume of a cube is 921 c.c. Find the area of one face of the cube.

5. Draw a pair of axes at right angles to each other. With reference to these axes, locate the points A, B, and C which have abscissas and ordinates as follows: —

Point	Abscissa	Ordinate
A	$+2$	$+7$
B	$+5$	-10
C	8	-4

6. How many cubic centimeters in the volume of a hemisphere of 8 cm. radius?

7. Draw three mutually rectangular axes of coordinates and locate the two points whose coördinates are $(-2, +3, +4)$ and $(+1, +5, +2)$.

8. Find the length of the vector joining the two points whose positions are given in the preceding problem.

9. Which of the two following points is farther from the origin, $(+5, -9)$ or $(+7, +7)$?

10. The two mutually perpendicular sides of a right-angled triangle are 9 and 20 centimeters, respectively, in length. Find the area of the triangle in square centimeters, and the length of the hypothenuse in centimeters.

11. Compute the number of seconds of arc in one radian.

12. Locate the following position vectors $\left\{\begin{matrix} r = 30, \theta = 15° \\ r = 30, \theta = 345° \\ r = 30, \theta = 195° \end{matrix}\right\}$.

13. Considered as a unit of volume, what advantage has the cubic centimeter over the cubic inch?

14. Indicate in three dimensions the position of a point whose polar coördinates are $r = 25$, $\theta = 45°$, $\phi = 90°$. See § 6.

15. Locate the two following points by use of polar coördinates and determine which of them is nearer to the line of reference, $(r = 12, \theta = 15°)$ and $(r = 6, \theta = 30°)$.

Change of Position

I. *Case of a Particle*

13. Since the position of a particle can always be defined by a straight line of a definite length drawn in a definite direction, it is evident that, if the particle moves to a new position, the new line which describes this new position will, in general, differ from the old one in its two characteristic features, viz. in length and in angular position. But this line may change in length only; in which case the particle moves along a straight line. It may change in angular position only; in which case we have motion on the surface of a sphere.

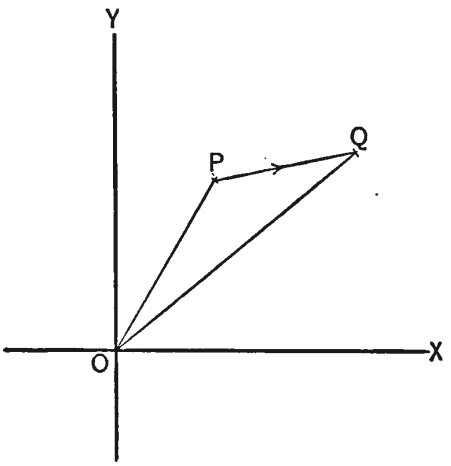


Fig. 6.— Change of position, a quantity having direction.

Suppose now that a particle at P (Fig. 6) has moved to a point Q such that the line $\overline{OP}$ has changed in length and also in angular position. If we take the old position of the particle P as a new point of reference, it is then clear that the *change* of position will be described by the straight line, $\overline{PQ}$, joining the new and old positions.

Evidently, then, we may describe the new position Q of the particle either by giving the two straight lines $\overline{OP}$ and $\overline{PQ}$, or by simply giving the one straight line $\overline{OQ}$. For, by either process, we definitely locate the new position Q.

The upshot of the whole matter is that change of position is described in exactly the same way as is position itself; namely, by stating the length, sense, and direction of a straight line.

Digression on the Addition and Subtraction of Vectors

14. Referring to Fig. 7, let us denote the vector $\overline{OP}$ by p, the vector $\overline{PQ}$ by $\bar{q}$, and the vector $\overline{OQ}$ by r. Then it will be clear that the new position Q of the particle may be located in either one of two different but strictly equivalent methods. By one method, the particle at Q is located as the terminal point of the

vector r. By the other method it is the terminal point of two vectors p and $\bar{q}$, placed end to end with their arrows in the same sense. **And since $\bar{r}$ locates the point Q at the same distance and in the same direction from the origin, $\bar{r}$ is said to be the resultant or vector sum of $\bar{p}$ and $\bar{q}$; while p and $\bar{q}$ are said to be the components of r.** In the algebra of vectors this fact is expressed by writing

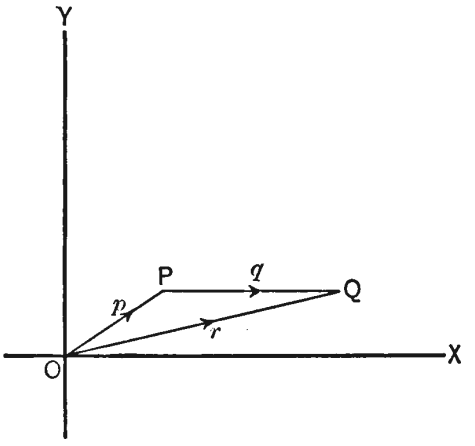


FIG. 7.—Vector addition.

$$p+q=r, \qquad \text{Eq. 1}$$

an equation which contains the whole story of vector addition. In ordinary algebra, which deals only with scalar quantities, we should have

$$p+q>r,$$

but this inequation refers merely to lengths and not at all to directions.

The resultant of any number of vectors is obtained by precisely the same process, namely, placing these vectors end to end, in any order, *but all in the same sense*, and then joining the initial point of the first vector to the terminal point of the last vector.

This graphical method gives rise to the vector polygon, as illustrated in Fig. 8, where r is the resultant of the vectors $\bar{p}$, q, and $\bar{s}$. From what precedes it will be seen that vector addition means nothing more than what is implied in the statement that, so far as mere transportation is concerned, it is all the same whether one travels by train directly from Chicago to Pittsburg, or goes first from Chicago to Toledo, thence to Cleveland, and thence to Pittsburg.

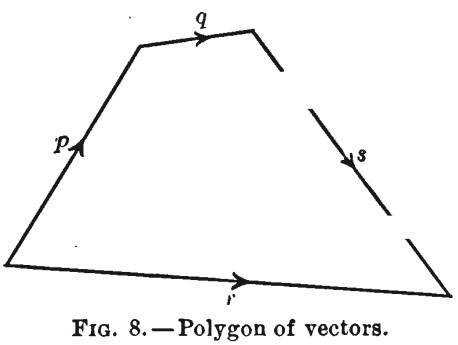


FIG. 8.—Polygon of vectors.

Since the sign of a vector determines its sense, one sees that the subtraction of any given vector from another is precisely the same as adding this vector to the other, having first reversed its sense.

When a particle receives a displacement which is represented by any given vector, say $\overline{OP}$, Fig. 9, the question often arises as to what the amount of the displacement is in some other direction, say $\overline{OH}$.

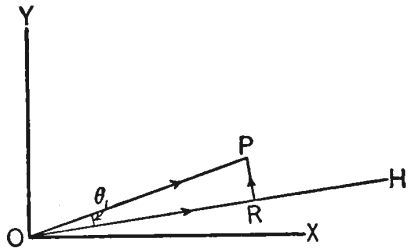


FIG. 9. — Illustrating the resolution of a vector.

This is found by letting fall from the terminal point P a perpendicular upon the line $\overline{OH}$. If R denote the foot of this perpendicular, we may consider the vector $\overline{OP}$ as broken up into two components $\overline{OR}$ and $\overline{RP}$, of which the former lies entirely along $\overline{OH}$; the latter has no component along $\overline{OH}$. From the definition of a cosine it is evident that

$$OR = OP \cos\theta. \qquad \text{Eq. 2}$$

II. *Case of a Rigid Body*

15. As has been observed (§ 7), the position of a rigid body is fixed when any three points in the body, not lying in the same straight line, are fixed. Accordingly, we see that the positions of bodies change in several *apparently* different ways.

(1) If only two points are fixed in the body, as, for instance, a wheel and axle held between two pivot points, AA' (Fig. 10), then the body is free to rotate about the line joining these two points. **The motion of a rigid body about a fixed axis is called a pure rotation.** A bicycle wheel lifted from the ground and spun is an instance.

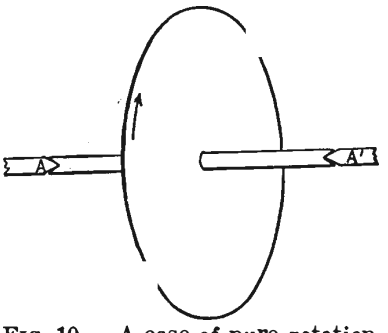


FIG. 10. — A case of pure rotation.

(2) It may happen that only one point is fixed in the body, as in the case of a walking-stick with its lower end on the ground. Here the body is free to rotate about *many* different lines as axes. It may, indeed, rotate about any straight line passing through the lower end of the stick.

(3) It may be that no point in the body is fixed, but that the body is compelled to move in such a way that *each point of the body describes a path equal and similar to the path of every other point.* Such a motion is called a **pure translation.** It

occurs in the piston rod of a steam engine. A sled sliding to and fro in its own tracks also represents the motion.

(4) Lastly, a body may be rotated *and* translated simultaneously. Such a case is represented in the wheel of a bicycle while being ridden.

The student who pursues this subject farther will find that a rigid body may be changed from any one position to any other position by simply translating it along a certain straight line and rotating it through a definite angle about a definite axis. Consequently the first three cases just considered are special cases of the fourth. The fact then is that every motion of any rigid body turns out to be either a pure translation or a pure rotation or a combination of these two; in other words, a rigid body may suffer a linear displacement or an angular displacement, or a combination of the two.

As we have already seen, linear displacements are represented by vectors; so also are angular displacements. For imagine the rough outline in Fig. 11 to represent any rigid body having one point O fixed. Take this point as origin. Imagine the body to be turned through any angle θ about an axis $\overline{OP}$. Such an angular displacement is what we have just called a pure rotation. It requires for its description three specifications, namely, (i) the direction of the axis of rotation, (ii) the amount of rotation, *i.e.* the angle through which the body has turned, and (iii) the sense of rotation. In short, this angular displacement requires a vector to represent it.*

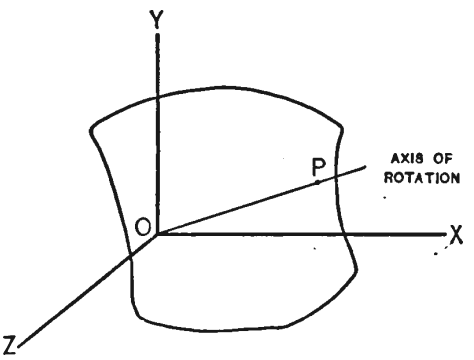


FIG. 11. — Angular displacement, a vector.

Thus, if a grindstone be rotated through 90°, its angular displacement would be represented by a vector drawn in the direction of the axis of the grindstone, and of a length $\frac{\pi}{2}$, the sense being such that if one looks in the direction of the arrow

* The student's attention is here especially directed to the fact that in adding vectors which represent angular displacement, the resultant is not independent of the order in which the vectors are added, except when the angular displacements are infinitesimal.

on the vector, the stone will appear to rotate in a clockwise direction. **Such a relation between the direction of the vector and the direction of the rotation is exactly that which exists between the direction of advance and the direction of rotation in an ordinary woodscrew. It is hence called the right-handed screw relation.** It is important to observe that the rule for the addition of vectors applies to angular displacements only when these are very small.

In the later parts of physics, we shall meet many quantities occurring in pairs similar to these two analogous displacements, linear and angular, which we have just been considering.

Degrees of Freedom

I. *Case of a Particle*

16. It is interesting to observe that when a particle is free to change its position by moving in any direction whatever, it can move in three, and only three, directions such that its motion along any one of these directions has no component along either of the other two. And it is evident that these three directions are mutually perpendicular ; for if they were inclined at any angle other than 90°, say θ, a displacement r along one axis would have a component $r \cos \theta$ along the other. These three directions are represented, therefore, by any three rectangular axes. In view of these facts, a particle is said to have three **degrees of freedom.** In other words, a particle has three possibilities in the way of translational motion. It cannot rotate because, by definition, it has no appreciable dimensions.

II. *Case of a Rigid Body*

17. The case is quite different with an extended body. Consider a coin, a metal cylinder, or any rigid body whatever ; if it is perfectly free, it can be translated parallel to any of the three rectangular axes. Like the particle, it has three degrees of translational freedom. Now imagine this body supported on a single pivot, in the same manner as a spinning top. Here one point is fixed so that the body has no freedom of translation ; yet it is clearly possible for the body to rotate about any one of three mutually perpendicular axes passing through this

fixed point. A rotation about any one of these axes will have no component of rotation about either of the other axes. A rigid body not constrained in any way is said, therefore, to have three degrees of freedom of rotation, making in all six degrees of freedom for any unhampered body, namely, three of translation and three of rotation.

The student will find it both interesting and important to discover, in any particular case of constrained motion, just how many degrees of freedom a body has; for the advanced student will find later that the whole behavior of a body may be described by use of as few coördinates as the body has degrees of freedom.

Interesting Special Case. Uniplanar Motion

18. When the motion is in a single plane, the rigid body can be transferred from any one position to any other position (in that plane, of course) by a pure rotation alone.

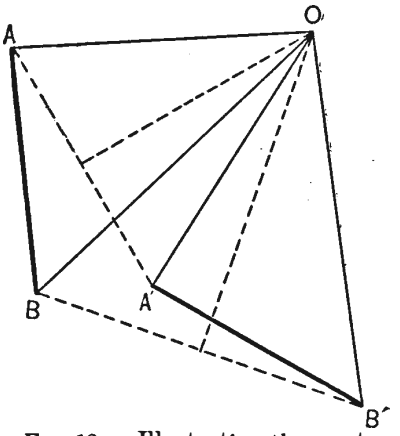


FIG. 12. — Illustrating the most general uniplanar motion.

To prove this theorem, consider any two positions of the body as represented by AB and $A'B'$ in Fig. 12. Bisect AA' and also BB'. Draw perpendiculars through these points of bisection. Denote by O the point where these perpendiculars intersect. Then the axis about which the body AB must be rotated to bring it into the position $A'B'$ is perpendicular to the plane, and passes through the point O. For evidently the triangle AOB may be rotated about this axis until it is exactly superposed upon the congruent triangle $A'OB'$.

Problems

1. How many degrees of freedom has a particle which is compelled to move in a plane?

2. A marble free to roll on a floor, but not free to leave the floor, has how many degrees of freedom?

3. Design a rigid body having one degree of freedom of translation and one of rotation.

4. How many degrees of freedom has one of the balls on the governor of a steam engine?

5. Most motions that are employed in actual machinery are constrained, *i.e.* they have less than six degrees of freedom. What, in your estimation, is the special case which occurs most frequently in machines?

6. In the following motions of rigid bodies distinguish clearly which are cases of pure translation, which are cases of pure rotation, and which are cases involving both translation and rotation: —

i. The motion of a door on its hinges.
ii. The motion of a carriage wheel with reference to its axle.
iii. The motion of a carriage wheel with reference to the road on which it travels.
iv. The motion of a rocking-chair as one rocks in it.
v. The motion of the eyepiece as one focuses a pair of opera glasses.
vi. The motion of the blade, with reference to the handle, when one opens a pocketknife.

7. What vector will describe the final position of a man who walks 10 miles N.W., then turns and walks 10 miles S.W.?

8. Two boats sail from the same port; at the end of 5 hours one has traveled 50 mi. due E., the other $\sqrt{5000}$ mi. S.E. What vector will then represent their relative positions? *Ans.* 50 miles due S.

9. The coordinates of two points are $(+2, +3)$ and $(-1, +4)$. Find the vector which joins this pair of points.

10. A particle moves in a circle starting from a given point which we may call the origin. Find the vector which describes the position of the particle when $\frac{2}{3}$ of the circumference has been traversed.

11. Plot the following four points and show that they are the corners of a parallelogram: $(+3, +2)$; $(+4, +1)$; $(+2, +5)$; $(+3, +4)$.

12. What are the rectangular components of the vector which joins the following two points: $(+7, +3)$; $(+10, +5)$?

Rate of Change of Position. Velocity

19. It is to be carefully observed that *up to this point* we have been concerned not at all with the history of the body between the time of its leaving its old position and the time of its reaching its new position. We do not care how long a body may have been in changing its position, nor does it matter through what "cause" it may have changed its position. We have been concerned only with a method for describing, most simply and most completely, the change in position of the body. When, however, it is asked *how* this change occurred, we are led to the consideration of *Time* and *Velocity*.

In the world about us we find motion to be a state of matter which is quite as natural — perhaps even more natural — than rest. Displacement may be considered as the result of motion. But very often we shall be concerned rather with the rate of motion. Accordingly, we must now consider the method of describing a rate of motion, *i.e.* a velocity.

When a man buys a railroad ticket from New Orleans to Boston, he pays for "displacement," and, in general, the price of a railroad ticket varies directly as the amount of displacement. But one's choice of a train depends largely upon its speed, or rate of displacement. The difference in price between a good watch and a poor one depends principally upon the rate at which the angular position of the hands change. If this rate is very uniform, the watch is well made; if the hands do not move at a fairly constant rate, the watch is poorly made.

While etymologically the same, the words "rate" and "ratio" now have different meanings; the former is a special case of the latter. For while "ratio" denotes the quotient of *any* one quantity by another, "rate" is employed in physics to mean only the ratio of some quantity to a *time*. Rate, unless otherwise specified, is always an exact synonym of time rate; time always appears in the *denominator* of any rate. The rate of interest is the amount of money one pays for the use of a dollar, divided by the number of years he has used the dollar. The death rate is the total number of deaths among a thousand people during any period, divided by the number of years in that period.

But before we can measure time rates we must decide upon a standard and a unit of time. These are described in the following paragraph.

Time. Unit of Time

20. The mean (i.e. the average) time occupied by the sun in one apparent revolution about the earth is known as the mean solar day. This interval is universally employed as the standard of time. **The $\frac{1}{86400}$ part of a mean solar day is called a second and is used by the scientific world as the unit of time.**

Intervals of time can be compared with marvelous accuracy by means of watches, clocks, or chronographs.

Definition of Speed

21. More than a century ago it was proved by Lavoisier, the great French chemist, that all matter has a continuous existence; from which fact it follows that, as a particle moves from one point to another, it must occupy every intermediate position along the route. This route is then a continuous geometrical curve. Accordingly the path of a particle is defined as the continuous locus of all the successive positions which it occupies. We are now able to define clearly what is meant by speed, namely, **the rate at which a particle moves along its path.** Direction of motion, it will be observed, does not enter into this definition. A man who, during fifteen minutes, walks a mile along a winding cowpath travels with precisely the same average speed as another man who walks a mile in fifteen minutes along a straight city pavement.

Let us denote by p the distance which a particle has moved along any path during the time t. Then for purposes of computation, the average speed of the particle is expressed as follows: —

$$\text{Average Speed} = S = \frac{p}{t}. \qquad \text{Eq. 3}$$

But if the speed of the particle is varying from point to point, as is usually the case in nature, then denote by p_1 the distance traversed by the particle at the end of the time t_1; and by p_2 the distance traversed by the end of the time t_2. Now if t_1 and t_2 be so chosen that t_1 is very little earlier than t; and t_2 a very little later than t, the speed at any instant t will be the limit approached by the ratio $\frac{p_2 - p_1}{t_2 - t_1}$. Accordingly one writes

$$\text{Speed at instant } t = S_t = \left[\frac{p_2 - p_1}{t_2 - t_1}\right]_{t_2 \doteq t_1} \qquad \text{Eq. 4}$$

where the subscript $t_2 \doteq t_1$ indicates that t_2 approaches indefinitely near to t_1.

The necessity for some expression of this kind will appear when one asks such a question as, "What is the speed of a baseball thrown vertically into the air?" The question has no meaning unless one specifies some particular instant of time; for the ball has every speed from the initial speed with which

it left the hand of the thrower to zero, which is its speed at the top of its path.

Speed is, therefore, really, a *limit*. The conception of a limit is something which each student should bring with him from a study of elementary geometry ; or, not having acquired it there, he should now master it once for all, since the idea is one which is essential even in elementary physics.

Unit of Speed

22. Having adopted the centimeter as the unit of length and the second as the unit of time, we have no trouble in settling upon the proper scientific unit of speed; namely, a speed of one centimeter per second. The engineer, however, generally employs the foot per second for his unit of speed.

Problems

1. Find the average speed, in miles per hour, of a steamer which makes the passage from New York to Liverpool, 3032 mi., in 6 da.

2. A 24-hour train from New York to Chicago, 926 mi., must make what average speed?

3. A shot travels a distance of 4 kilometers in 10.5 sec. Find its average speed in meters per second.

4. A speed of 1 ft. per second is equivalent to a speed of how many centimeters per second?

5. A gun is fired at a distance of 1 kilometer; the report is heard 3.25 sec. later. From this observation, compute the speed of sound in air.

6. Light travels from the sun to the earth in about 8 min. The distance of the sun from the earth is about 95 million mi. Find the approximate speed of light in miles per second. What is the value of this speed in meters per second?

7. One man walks at the rate of 3 mi. per hour; another walks at the rate of 7 km. per hour. Which one travels faster?

8. Assuming that the circumference of the earth at the equator is 25,000 mi., find the speed of a particle on the equator rotating once in 24 hours. Express this speed in miles per second, also in meters per second.

Definition of Velocity

23. It is evident from the preceding examples that speed alone is not sufficient to determine the rate at which a particle is changing *position*. Consider, for instance, a particle which

moves through a small distance from P_2 to P_3 along the curved path indicated in Fig. 13, where P_2 is located by the vector r_2 and P_3 by the vector r_3. In order to describe the rate of change of *position*, it is necessary to know not only how far the particle moved in passing from P_2 to P_3, but also in what direction the line joining P_2 and P_3 lies.

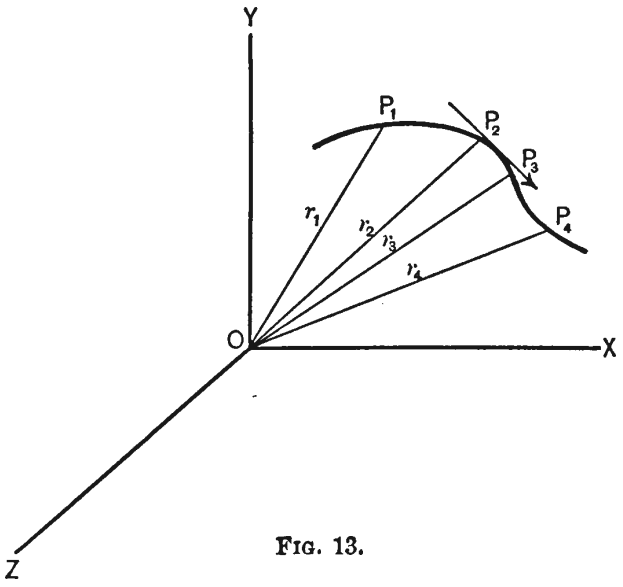


FIG. 13.

It is clear, of course, that when a particle moves, it always moves at any particular instant, in some one direction, *and* in some one sense. To completely describe the motion at any instant it is necessary, therefore, to know not only the speed but also the direction and the sense of the motion at that instant. This can be most conveniently done by use of **a vector called "velocity" which has the same direction and sense as that of the motion and which has a length equal to the speed of the motion.** Such a vector completely describes the velocity of the particle at the particular instant under consideration. Velocity is, therefore, a vector quantity, and speed a scalar. To illustrate: "A boat leaves her pier and steams in a straight line at a speed of 10 miles an hour for 3 hours." From this information, all that we know of the boat's position is that she now lies somewhere on a circle of 30 miles' radius, having the wharf for center. If, however, her velocity (instead of her speed) had been given by saying that "she steamed for 3 hours at a 10-mile rate in a northeasterly direction," her present position would be accurately known. **Velocity** is, accordingly, defined as the **time rate of change of position** of the particle. Or, in terms of algebra, if we denote velocities by V,

$$V = \left[\frac{\overline{P_1P_2}}{t_2 - t_1}\right]_{t_1 \doteq t_2} \qquad \text{Eq. 5}$$

Note carefully the difference between this equation, which defines velocity, and Eq. 4, which defines speed ; the numerator in (5) is a vector, while the numerator in (4) is a scalar. From this follows the

Distinction between Speed and Velocity

24. Speed represents merely the rapidity of motion, while velocity represents rapidity, direction, and sense. Speed is defined by a single number, while velocity demands three specifications ; namely, amount, direction, and sense. Imagine two trains traveling along the same piece of double track, the one going north 30 miles an hour, the other going south 30 miles an hour. Both trains have the same speed and the same direction ; but their velocities are as different as they can be ; for they are directly opposite in sign, *i.e.* in sense.

Velocity of a Body

25. We leave now the discussion of a particle and pass to the consideration of an extended body, *i.e.* of a body whose dimensions cannot be neglected in comparison with other quantities involved. And since we have found (§ 15) that any change in the position of a body can be defined by not less than two vectors, a translation and a rotation, it follows that the *rate* at which a body changes position will require for its description two velocities.

If the motion of the body is one of translation only, it is the same as that of a car along a straight piece of railroad ; the linear velocity is the same for every particle in the car ; and this velocity is completely described when the speed of any one point on the car and the direction of the track are given.

But suppose the car is placed on a turntable (Fig. 14), and the table then made to rotate ; particles near the end of the car will move rapidly, those near the center will move slowly. In general, the speed of a particle will vary as the distance from the axis of the turntable. How shall we describe a motion which is apparently so complicated ? This is very easily done when we recall that practically the only quantity changing is the angular position of the car, and that this change is the same for every part of the car.

Imagine this car to have turned through an angle of θ radians.

Consider next any *horizontal* line drawn in the car, the back of a seat, or the edge of a window sill; each has been turned through the same angle θ. Indeed, the new position of the car is definitely fixed, when the motion is one of rotation only,

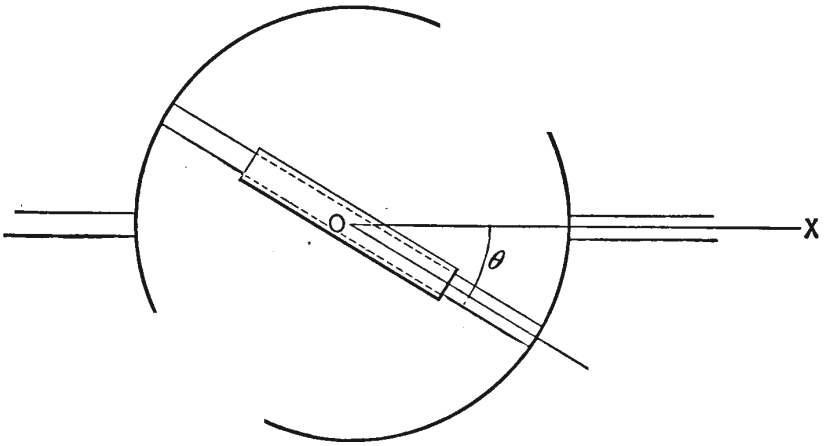


FIG. 14.

by saying that *the axis of rotation is vertical, the amount of rotation,* θ. The rate at which the angle θ varies with time is called the **angular speed** of the car (regardless of the axis about which the rotation takes place).

ANGULAR SPEED

26. And, in general, if θ_1 be the angular position of any line in a body at a time t_1, and θ_2 its angular position at time t_2, then its average angular speed ω is defined and completely described by the following equation:

$$\omega = \frac{\theta_2 - \theta_1}{t_2 - t_1}$$ Defining equation for *average* angular speed. Eq. 6

Unit angular speed is evidently that with which a body must rotate in order to describe one radian per second.

The engineer generally employs a unit of angular speed which is only about one tenth as large as this, namely, *one revolution per minute;* for which he uses the symbol "R.P.M."

If, instead of the average angular speed, we wish to measure the actual angular speed at any particular instant, we must then take the average speed over an indefinitely small interval of

time, including this particular instant. This operation is indicated in the following notation :

$$\omega = \left[\frac{\theta_2 - \theta_1}{t_2 - t_1}\right]_{t_1=t_2}$$ Defining equation for angular speed *at any instant.* Eq. 7

Angular Velocity

27. The numerical value of ω (in Eq. 7) together with the direction and sense of the axis about which the rotation occurs are the three specifications which describe any angular velocity. It is evident, then, that one specification is sufficient to define angular speed, but three specifications are required to define angular velocity.

The graphical representation of angular velocity is quite as simple as that of linear velocity; for we have only to lay off along the axis of rotation as many units of length (centimeters) as there are radians per second in the angular speed.

With this convention, the straight line $\overline{OX}$ (Fig. 15) completely represents an angular velocity; for its length measures the angular speed and its direction is the direction of the axis of rotation, while the arrow indicates the sense of rotation.

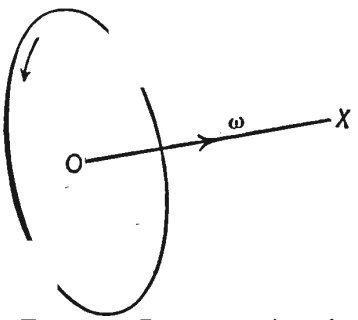


Fig. 15. — Representation of angular velocity by a straight line.

An angular velocity is said to be positive when, on looking along the positive direction of the axis OX, the body appears to rotate in the same sense that a right-handed screw appears to rotate when one looks along the direction of its advance. When the rotation is in the opposite sense, it is called negative.

The rate at which the angular velocity of the earth is changing is fairly well known; for its axis changes direction at a constant rate and describes a complete circle in the sky once in every 25,000 years. No variation in the angular *speed* of the earth has, however, certainly been detected; although it is well known that the friction of the tides must be slowing up the earth to some slight, though as yet unmeasurable, extent. This phenomenon is beautifully illustrated by spinning any ordinary top. From Eqs. 4 and 6 it is clear that angular speed

is a quantity which is strictly analogous to linear speed; while Eqs. 5 and 7 show that linear and angular velocities are likewise strictly analogous.

Relation between Linear and Angular Speed

28. Consider any point P (Fig. 16) in a rigid body rotating about any fixed axis, a wheel on its axle, for instance. Let the perpendicular distance of this point from the axis of rotation be r. If the angular speed of the body is ω, it will rotate once in $\frac{2\pi}{\omega}$ seconds.

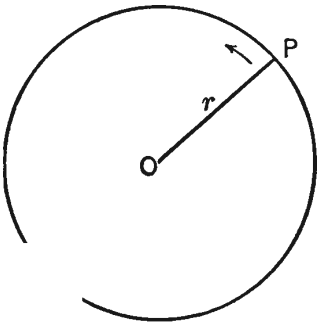


Fig. 16.—Relation between linear and angular speed.

But while the body performs one rotation, the particle at P has moved over a distance of $2\pi r$.

Its average *linear* speed is, therefore,

$$\frac{\text{Distance}}{\text{Time}} = \frac{2\pi r}{\left(\frac{2\pi}{\omega}\right)} = r\omega = S.$$

In the case of a fixed axis, the angular speed of the body, and the linear speed of any particle on it, are, therefore, related by the following equation: —

$$S = r\omega. \qquad \text{Eq. 8}$$

Problems

1. A boy attaches a small stone to the end of a string which is 80 centimeters long, and then swings it about his head at the rate of 3 turns per second. What is the linear speed of the stone in centimeters per second?

2. The belt of a driving pulley travels at a speed of 15 m. per second. The pulley rotates 10 times per second. Assuming that the belt does not slip on the face of the pulley, compute the diameter of the pulley.

Ans. $\frac{75}{\pi}$ cm.

3. In the case of a watch, compute the angular speeds of the second hand and the hour hand, respectively.

Ans. $\frac{\pi}{30}$ and $\frac{\pi}{21600}$.

4. What is the angular speed of a bicycle wheel making 30 revolutions per minute? *Ans.* π radians per second.

5. The front wheel of a bicycle is 28 in. in diameter. Find the angular speed of the wheel about its axle when it is ridden at the rate of 12 mi. per hour. *Ans.* 15.09 radians per second.

6. How may the ordinary peg top be used to illustrate the case of a body having a constant angular speed, but at the same instant a variable angular velocity?

7. Show that the fly wheel of an engine, when the steam is cut off, illustrates an angular velocity which is uniform in direction, but variable in speed.

8. What is the aim of the clockmaker, to produce an instrument which will give constant angular speed or constant angular velocity?

CHANGE OF VELOCITY. ADDITION AND SUBTRACTION OF VELOCITIES

29. Up to this point we have considered a particle which is urged only by one single velocity. It often happens, however, that a particle has impressed upon it two or more different velocities at the same time. For instance, if a man be sitting in a moving street car, his motion with respect to the earth is directly along the rails. If, however, he change his seat to one directly opposite, on the other side of the car, his motion with respect to the earth, as he walks across the car, is not along the rails, nor across the rails, but in a diagonal direction, $\overline{AC}$, as indicated in Fig. 17. $\overline{AB}$ is a straight line which indicates the

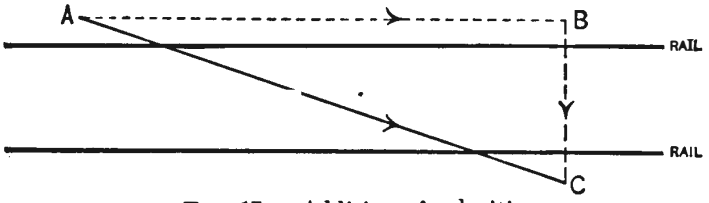


FIG. 17. — Addition of velocities.

velocity of the car; $\overline{BC}$ is a straight line which represents the velocity of the man with respect to the car. These vectors are added as are all other vectors, and $\overline{AC}$ is said to be the *resultant velocity* of the *components* $\overline{AB}$ and $\overline{BC}$. $\overline{AC}$ is, therefore, the straight line which represents the man's motion with respect to the earth.

Since velocity is a vector quantity, it is evident that we may at once apply to it the general rules for vector addition and subtraction. For these see § 14 above. Thus having given a particle whose speed is v and whose direction is defined by θ, its velocity will be the vector $\overline{OP}$, in Fig. 18. Let us suppose the motion to take place in the $\overline{XY}$ plane. Then if we desire

the velocity of the particle in a direction parallel to the axis of X, it is only necessary to project $\overline{OP}$ on the axis of $\overline{OX}$. We thus obtain $\overline{OH}$ as the required velocity. In like manner $\overline{HP}$ will be the velocity parallel to the axis of Y; or in terms of algebra

$$\left.\begin{aligned} \overline{OH} &= \overline{OP}\cos\theta, \\ \overline{HP} &= \overline{OP}\sin\theta, \\ \overline{OH}^2 + \overline{HP}^2 &= \overline{OP}^2. \end{aligned}\right\} \qquad \text{Eq. 9}$$

In like manner, if it is desired to find the component of the velocity $\overline{OP}$, resolved in any other direction, say, along the line $\overline{OR}$, which makes an angle ϕ with $\overline{OP}$, we have

$$\overline{OR} = \overline{OP}\cos\phi. \qquad \text{Eq. 10}$$

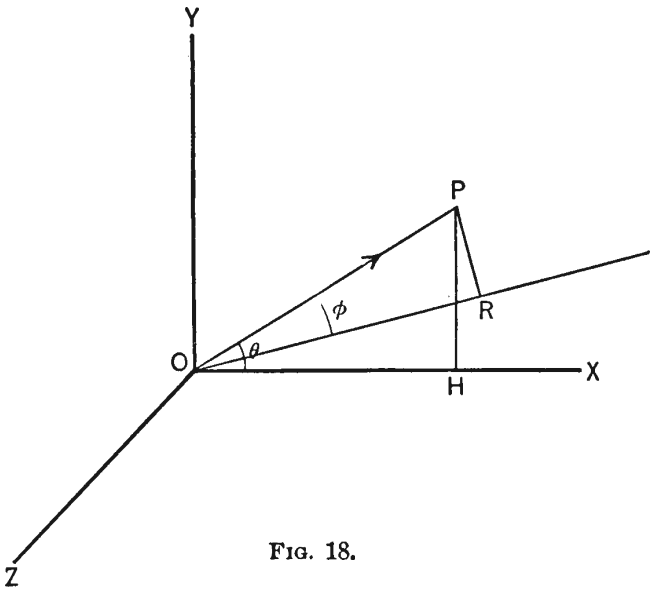


FIG. 18.

This same result may be expressed graphically by saying that the component of a velocity in any given direction is the projection of the vector which represents that velocity upon a straight line drawn in the given direction. Angular velocities are compounded and resolved in precisely the same manner as linear velocities.

ALL MOTION RELATIVE

30. In the case above considered, where a man changes his seat from one side of a car to the other, the question might have been asked: What is his motion *with respect to the car?* The answer would then have been that his velocity with respect to the car is completely represented by the straight line $\overline{BC}$, Fig. 17.

Whether, then, the man's velocity is represented by $\overline{AC}$ or $\overline{BC}$, depends entirely upon whether we take the earth or the car as our reference point.

And so it is with all velocities. Each one is a velocity with reference to some point which for the time being we consider as fixed. If a man be seated in a railway car which is traveling at the rate of 30 miles an hour, we may say that the man is at rest or in motion, according as we take for our reference point some part of the car or some part of the earth.

Problems

1. Represent by a diagram the velocity of a boat which is being rowed east, at the rate of 4 mi. an hour, while the current is carrying it south at the rate of 1 mi. an hour.

2. A wheelman is riding north at the rate of 8 mi. an hour against a head wind which is retarding him at the rate of 2 mi. an hour. What speed would he make if there were no wind?

3. What speed would this same wheelman make if he turned around and rode with the wind?

4. A canal boat is towed at the rate of 3 mi. an hour. How fast must a man walk its deck in order that he may remain at rest with respect to the towpath?

5. A wheel rider travels east at the rate of 6 mi. per hour while the wind is blowing from the north with a speed of 6 mi. per hour. From what direction will the wind appear to strike the rider?

6. A boat is rowed on the river so that its speed in still water would be 6 mi. an hour. Suppose the river flows at the rate of 4 mi. an hour. Make a diagram showing the direction in which the boat must head in order that its motion with reference to the bottom of the river may be at right angles to the current.

7. A wheel having a diameter of 28 inches revolves once in two seconds. Find the distance traversed in one hour by a particle on the circumference.

8. Two trains of the same length are running with the same speed on parallel tracks, but in opposite directions. Their combined length is 800 ft., and they pass each other in 6 sec. What is the velocity of the trains relative to the track? From Shearer's *Notes and Questions in Physics*, No. 49.

9. An express train runs due north, from Chicago to Milwaukee, at the rate of 50 mi. an hour, during a west wind which is blowing at the rate of 25 mi. an hour. Find the direction of the smoke which leaves the locomotive.

10. Find the resultant of the following three velocities, 10, 6, and $10\sqrt{2}$, which make the angles + 45°, 0°, and − 30° respectively with the horizontal.
Ans. 24.2 in a horizontal direction.

11. Two velocities each of 16 cm. per second include an angle of 120° between them. Find the resultant.

Rate of Change of Velocity. Acceleration

31. Having now considered three fundamental ideas (position, change of position, and velocity), we proceed to the study of a fourth fundamental quantity; namely, rate of change of velocity.

If all the motions we encounter in nature had velocities which were uniform both in direction and in speed, the vocabulary of physics would be even smaller than it now is. Although the behavior of a good clock or the rotation of the earth on its axis are marvelously close approximations to constant angular speeds, it is an interesting fact that no instance of perfectly constant speed has yet been discovered in nature or invented by man. The motion of a cutting tool on a shaper is fairly constant so far as direction is concerned; but its speed is by no means constant.

Definition of Linear Acceleration

32. If the velocity of a particle is not uniform, it must be because either the direction of the motion, or the speed of the motion, is changing, or both. For these are the only two continuously variable elements in velocity. When either or both of these changes occur, *i.e.* when the velocity of the particle is changing, the motion is said to be accelerated (*ad+celero*).

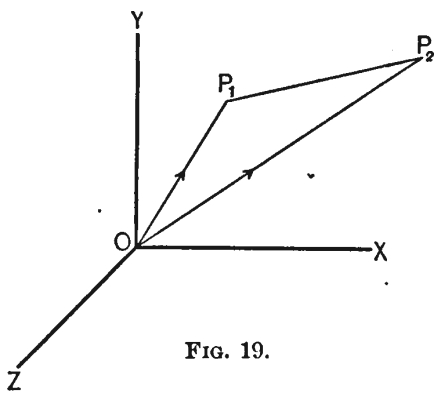


Fig. 19.

If the vector $\overline{OP}_1$ (Fig. 19) represents the velocity of the particle at a time t_1, and the vector $\overline{OP}_2$ represents it at a time t_2, we know from the properties of vectors that the total change in velocity during the interval $t_2 - t_1$ is represented by the vector $\overline{P_1P_2}$. In other words, the straight line joining P_1 and P_2 is as many centimeters long as there are units of speed in the

increment of velocity $\overline{P_1P_2}$. Not only so, but the direction of this straight line is the same as that of the added velocity.

To the rate, $\frac{\overline{P_1P_2}}{t_2 - t_1}$, has been given the name **acceleration.** Let us denote this acceleration by a, the change in velocity by v, then

$$a = \frac{\overline{P_1P_2}}{t_2 - t_1} = \frac{v}{t_2 - t_1}. \quad \text{Eq. 11}$$

Defining equation for *average* acceleration.

Acceleration, then, is simply the rate at which the particle is gaining or losing velocity. This rate is not always constant; consequently, the average acceleration will depend in such a case upon how long an interval, $t_2 - t_1$, is covered by the average. In order to get the actual acceleration at any particular point of time t, we must take the average over an indefinitely small interval of time, including t.

$$a = \left[\frac{\overline{P_1P_2}}{t_2 - t_1}\right]_{t_2 \doteq t_1} = \left[\frac{v}{t_2 - t_1}\right]_{t_2 \doteq t_1} \quad \text{Eq. 12}$$

Defining equation for acceleration *at any instant t.*

Acceleration is accordingly the limit approached by the ratio between an increment of velocity and the corresponding increment in time. If, therefore, a particle is moving with uniform velocity, its acceleration is zero. Whether the particle is moving fast or slowly makes no difference. So long as its velocity is not changing, its acceleration is zero.

Unit of Acceleration

33. From this "defining equation," our unit of acceleration is obtained exactly as we obtained our unit of velocity, viz., we make each of the terms in the right-hand member unity, then a must be unity. We thus arrive at the following definition: **Unit acceleration is an acceleration in which unit change of velocity is produced in unit of time.**

Since, in pure science, the centimeter per second is the unit of velocity, the natural unit of acceleration is the *centimeter per second per second*, often written cm./sec.^{2}. Thus a projectile fired from a gun gains during each second a velocity of approximately 981 centimeters per second toward the center of the earth. Accordingly the acceleration of a body falling freely under gravity is said to be 981 cm./sec.^{2}. In engineers' units this quantity would read approximately 32.2 ft./sec.^{2}.

Angular Acceleration

34. Angular velocity, like linear, depends for its value upon two variables, namely, the direction of the axis of rotation, and rate of rotation. If, in any spinning body, either of these two variables changes, the body is said to have an angular acceleration. Thus the shaft of a turbine steamer may be rotated with a constant angular speed, but if the steamer is changing her course, the angular velocity of the shaft is changing, and the shaft is said to have an angular acceleration. If, at another time, the boat be moving in a perfectly straight line and the speed of the turbine be varying, the motion of the shaft is also said to have an angular acceleration.

In general, the velocity of a spinning body varies from instant to instant; as, for example, the fly wheel of a stationary engine; so that if we wish to obtain the angular velocity at a particular instant t, we must choose t_2 and t_1 in such a way that t_2 shall be a time a very little later than t, and t_1 a time a very little earlier than t. Then, if we denote by ω_1 the angular velocity at the instant t_1, and by ω_2 the angular velocity at the instant t_2, the angular acceleration at any instant t will be the limit approached by A as $t_2 - t_1$ approaches zero.

$$A = \left[\frac{\omega_2 - \omega_1}{t_2 - t_1}\right]_{t_2 \doteq t_1} \qquad \text{Defining equation for angular acceleration.} \qquad \text{Eq. 13}$$

Since the numerator in the right-hand member is a vector quantity, while the denominator is a scalar, it follows that the angular acceleration A is also a vector quantity, and is, therefore, compounded and resolved as are other vector quantities.

Two Important Special Cases

35. Having now considered in some detail the three fundamental conceptions of kinematics, namely position, velocity, and acceleration, we proceed to the study of two highly interesting special cases.

Case I. *Motion of a Particle along a Straight Line: Direction of Velocity Constant; Speed Variable*

36. If, in Fig. 20, the velocity of a particle be changed from $\overline{OP_1}$ to $\overline{OP_2}$ (the points O, P_1, P_2 all lying on the same

straight line), then the acceleration results from a change of speed only. For the velocity of the particle is all the while along the straight line $\overline{OP}_2$. Here the acceleration is in the same direction as the motion of the particle.

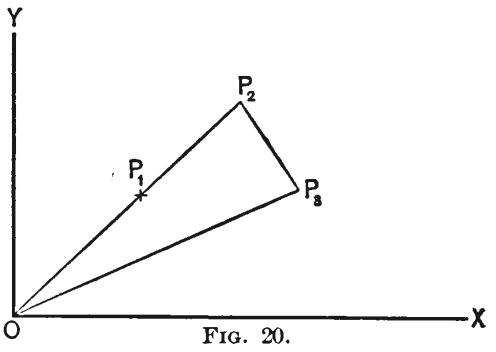


FIG. 20.

Let us imagine that the speed of a particle changes from OP_1 to OP_2 in any interval of time t, and that, throughout the motion, *the acceleration is constant.* Let us call the acceleration a; then the particle will gain a units of speed during each second. The *total gain of speed* during any time t is therefore given by the following expression: —

$$s = at. \qquad \text{Eq. 14}$$

If, now, we happen to know the speed of the particle at the beginning of the interval and call it S_0, then the actual speed at the end of the interval will be the original speed plus the speed gained, viz.: —

$$S = S_0 + s,$$

or

$$S = S_0 + at. \qquad \text{Eq. 15}$$

This equation is very useful; for it not only describes the variable speed S at each instant of time, but it also enables us to compute the value of any one of the four quantities involved as soon as we know the other three.

In case a particle loses speed, then we have a negative value for s, and hence a negative value for a in the equation, $a = -\frac{s}{t}$. For in these problems, t is always positive.

Not infrequently we shall want to know something more about the motion of the particle than its mere speed. We often want to know what the position of the particle is after a certain interval during which its motion has been accelerated.

If the acceleration a is a constant, then the gain in speed as time goes on will be proportional to the time,

$$s = at.$$

And the *mean* speed of the particle from the beginning to end of the interval is the value of the speed at the middle of this

interval, viz. the speed represented by the line LM, in Fig. 21, but this speed is, by definition, the mean of S_0 and $S_0 + at$, which is $S_0 + \frac{at}{2}$. Call this average speed, S. We may now take this average value of the speed and substitute it for S in Eq. 3, § 21, thus obtaining

$$p = St = \left(S_0 + \frac{at}{2}\right)t,$$

or

$$p = S_0 t + \tfrac{1}{2} at^2. \qquad \text{Eq. 16}$$

where p is the length of path (or simply the distance) covered by the moving particle.

For the special case which we are considering, namely, motion in a straight line, this equation, together with Eq. 15, $S = S_0 + at$, tells the whole story. One gives us the speed of the particle when we know the acceleration and the duration of its action; the other gives us the change in the position of the particle.

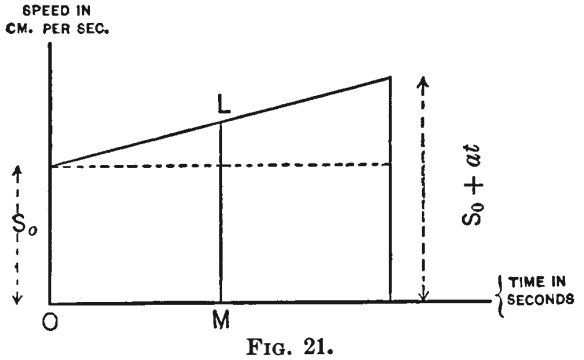


FIG. 21.

If between these two equations we eliminate t, we get another useful expression giving us the speed of a particle in terms of its acceleration and the distance traversed, namely

$$S^2 = S_0^2 + 2ap. \qquad \text{Eq. 17}$$

To solve any problem, then, under this special case which we have just been considering (direction of velocity constant: speed variable), one has merely to observe which of the above five quantities (S, S_0, a, t, and p) are given, which required, and lastly which of these three equations contains the desired and the known quantities. He has then one equation to solve for the unknown quantity.

Problems

1. A bicycle and rider start from rest at the top of a hill. At the end of 16 seconds the foot of the hill is reached, with a speed of 640 cm. per second. Find the average acceleration of the wheel during the descent of the hill. Here S, S_0, and t are given. *Ans.* $a = 40$.

2. Having computed the acceleration, use this and the speed at the foot of the hill to compute the length of the hill. *Ans.* $x = 51.2$ m.

3. A train running at 36 km. per hour is stopped by a sudden application of the brake. What acceleration must be produced by the brakes in order to stop the train in 8 sec.? Here, again, S, S_0, and t are given.

Ans. $a = -125$.

4. When steam is turned on again, a constant acceleration of 50 units is produced. How long will it be before the train has again acquired its original speed of 36 km. per hour? *Ans.* 20 sec.

5. Over what distance will this train have traveled while coming to rest in the third problem? *Ans.* 40 m.

6. How far will the train travel while again acquiring its original speed in the fourth problem? *Ans.* 100 m.

7. A football player running north at the rate of 8 m. per second reverses his velocity and in an interval of 2 sec. is again running, this time south, with a speed of 8 m. per second. Find his mean acceleration during the interval. *Ans.* $a = -800$.

8. A body is simultaneously urged to move with speeds of 40, 15, and 20 respectively. Can it remain at rest?

9. A train which is uniformly accelerated starts from rest, and at the end of 3 sec. has acquired a speed with which it would travel through 1 km. in the next 5 min. Find the acceleration.

10. Two bodies, whose respective velocities are accelerated in every second by 30 and 50 cm. per second, begin to move toward each other at the same instant. At first they are 2 km. apart, and have no initial velocity. After how many seconds will they meet?

11. A body tends to move with equal speeds in two directions inclined to each other at 120°. Find its path and resultant velocity.

12. One man A is standing at a point 1500 ft. east of another man B. At the same instant both men start to walk east, B at the rate of 4 mi. an hour, and A at the rate of 3 mi. an hour. How long will be required for B to overtake A?

13. A particle has an initial speed of 20 cm. a second, and receives an acceleration of 8 cm. per second every second. What distance will it travel in 15 sec.?

14. A wind is blowing from a point intermediate between north and east. The northerly component of its velocity is 10 mi. an hour, and the easterly component is 36 mi. an hour. Find the whole velocity.

15. Write in the margin of your book the special form which Eqs. 15, 16, and 17 assume when the particle starts from rest.

CASE II. *Uniform Motion in a Circle: Direction of Velocity Variable; Speed Constant*

37. If, however, in Fig. 22, the velocity of a particle be changed from $\overline{OP}_2$ to $\overline{OP}_3$, where the vector $\overline{OP}_3$ is constantly equal in length to $\overline{OP}_2$, *there is no change in speed; but there is a change in the direction of the velocity. The motion is, therefore, accelerated.*

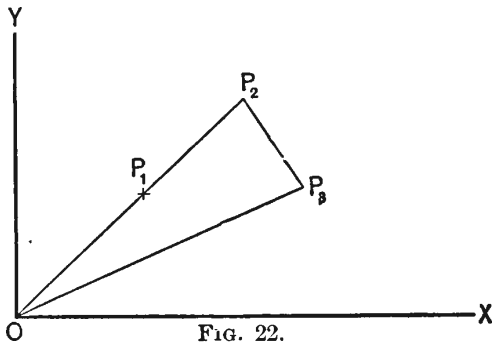


FIG. 22.

Such a motion takes place when a particle moves in a circle at a uniform speed. It is approximated in the following instances: when a boy whirls about his head a small mass attached to the end of a string; a small particle of gravel sticking to the wheel of a bicycle in motion; the earth in its revolution about the sun; the motion of the moon in its orbit about the earth; each particle which goes to make up the fly wheel of an engine.

In each of these cases, let us consider the center of the circular path O (Fig. 23) as a fixed point of reference. And let us draw a straight line from O through the point where the particle lies when we first begin to observe its motion. Call this line $\overline{OX}$.

VELOCITY VECTOR

For a clear understanding of this motion, it is next essential that the student distinguish carefully between the position, the path, and the velocity of the moving particle.

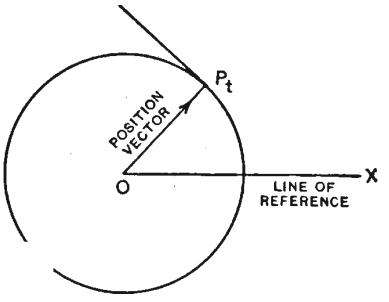


FIG. 23.

The position of a particle P_t at any instant t is completely given when we know the particular radius which joins the particle and the center O at that instant.

By the path of a particle is meant (§ 21) the line which is made up of the successive positions of the particle.

The velocity of the particle at any instant is evidently in a direction which is tangent to the circle at the point where the particle is at that particular instant.

Imagine the particle at any time to be at the point P_t in Fig. 23. The direction of the velocity will there be perpendicular to the radius $\overline{OP_t}$. And the same is true at any other instant, for in circular motion the straight line which represents the velocity and the radius which represents the position of the particle are always at right angles to each other.

If we wish to represent the velocity completely, we must choose for the constant length of this tangent line as many centimeters as there are units in the constant speed of the particle.

In Fig. 24 let P_1, P_2, P_3, etc., represent positions of a particle P at small intervals of time t. Imagine the particle to be moving with uniform speed in a circle. $\overline{OV_1}$, $\overline{OV_2}$, $\overline{OV_3}$ (drawn at right angles to $\overline{OP_1}$, $\overline{OP_2}$, $\overline{OP_3}$, respectively) may then represent the *velocities* of the particle in its positions P_1, P_2, P_3, respectively.

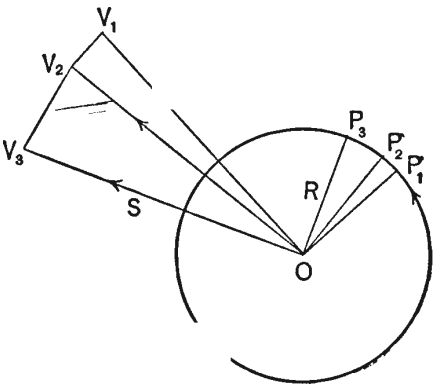


FIG. 24.

We can now determine at once the acceleration of the particle. For, if the angular speed of the radius $\overline{OP_1}$ be ω, so that the speed of the particle P is $S = \omega R$, then the angular speed of $\overline{OV_1}$ is also ω, because it keeps always just 90° ahead of $\overline{OP_1}$; and the point V_1 will describe a circle of radius S, with a speed ωS. Now the total change in the velocity of the particle P — remember there is no change in speed — during the time t is completely represented by the line $\overline{V_1V_2}$. For $\overline{V_1V_2}$ is the velocity which must be added to the velocity $\overline{OV_1}$ to give the velocity $\overline{OV_2}$. And hence, by definition,

$$\text{Acceleration of the particle} = \frac{\overline{V_1V_2}}{t}.$$

But $\dfrac{\overline{V_1V_2}}{t}$ measures the speed of the point V, and is hence equal to ωS, in a direction at right angles to $\overline{OV}$.

Hence the acceleration of a particle moving with speed S in a circle is completely described by saying that the direction of the acceleration is parallel to $\overline{PO}$, and the amount of the acceleration is equal to ωS. Or we may say the **acceleration is always toward the center and is numerically equal to ωS.**

And since (Eq. 8, § 28) $S = \omega R$, we may also write

$$\left.\begin{matrix}\text{Acceleration of a particle} \\ P \text{ moving in a circle}\end{matrix}\right\} = \omega S = \omega^2 R = \frac{S^2}{R}. \quad \text{Eq. 18}$$

If we denote the time of one revolution of a particle in a circle — an interval known as the **period** of the motion — by T, then we have $T = \frac{2\pi}{\omega}$; and hence substituting for ω, in Eq. 18,

$$\text{Acceleration towards center} = \frac{2\pi S}{T} = \frac{2\pi R\omega}{T}.$$

We have now a complete description of the motion of a particle which moves with uniform speed in a circle. The student should be able to put into words the meaning of each of these different expressions. For example, the last form, $\frac{S^2}{R}$, says that "the acceleration of a particle moving with uniform speed in a circle is constant, and is numerically equal to the linear speed of the particle squared and divided by the radius of the path."

General Case of Linear Acceleration

38. We have now considered, in some detail, two special cases of acceleration, in one of which the speed alone varies, in the other of which the direction alone varies.

The importance which attaches to these two cases arises partly from the fact that the acceleration of any particle whatever, moving at any given instant in any path, however tortuous, may always be resolved into two components, one along the path, and one normal to the path; and these two components belong respectively to the two special cases which we have just studied. We thus have the general result: —

$$\left.\begin{matrix}\text{Total acceleration of} \\ \text{any particle}\end{matrix}\right\} = \left\{\begin{matrix}\text{Vector sum of the acceleration along} \\ \text{the path and the acceleration normal} \\ \text{to the path.}\end{matrix}\right.$$

The advanced student will discover in Vector Analysis a much more elegant method of treating this subject.

Addition and Subtraction of Accelerations

Since acceleration is a vector quantity, it follows at once that accelerations are added and subtracted in the same manner as velocities and other vectors. See § 14.

Definition of Angular Acceleration

39. Consider any spinning body, such as the propeller shaft in a steamer, a top, the earth, the fly wheel of an engine, or a grindstone. It is at once clear that the angular velocity with which any of these rigid bodies is rotating may change in either one of two ways, as already intimated in § 34, namely,—

(i) The angular speed of the body about its axis may change (as, for instance, that of a fly wheel when the engine is just starting or stopping), and

(ii) The direction of the axis of rotation may change (as, for instance, that of the propeller shaft, when the steamer is turning around without slowing up her engines). A body which is changing in either one of these two respects is said to have an **angular acceleration.** This quantity is a vector and a strict analogue of linear acceleration. Denoting angular velocities by ω_1 and ω_2, and angular acceleration by A, we have already (§ 34) obtained the defining equation

$$A = \left[\frac{\omega_2 - \omega_1}{t_2 - t_1}\right]_{t_1 = t_2} \qquad \text{Eq. 13}$$

Unit of Angular Acceleration

40. **When the rate at which a body is rotating changes by one radian per second during each second, the value of its angular acceleration is unity.**

Two Important Special Cases

41. Corresponding to the two special cases of linear acceleration, we have here also two special cases.

Case I. *Direction of Axis of Spin Constant; Rate of Spin Variable*

42. Here the whole story is told by three simple equations which are identical in form with Eqs. 15, 16, and 17, § 36. Denoting angles by θ, angular speeds by ω, angular acceleration by A, and time by t, one has

$$\omega = \omega_0 + At, \qquad \text{Eq. } 15'$$

$$\theta = \omega_0 t + \tfrac{1}{2} A t^2, \qquad \text{Eq. } 16'$$

$$\omega^2 = \omega_0^2 + 2A\theta. \qquad \text{Eq. } 17'$$

CASE II. *Rate of Spin Constant; Direction of Axis of Spin Variable*

43. Here again the analogue is complete; the axis about which the acceleration occurs is at right angles to the axis of spin; and the amount of acceleration is given by an expression which is identical in form with Eq. 18, § 37, namely,

$$A = \left\{\begin{matrix}\text{Acceleration about}\\ \text{axis normal to axis}\\ \text{of spin}\end{matrix}\right\} = \Omega\omega, \qquad \text{Eq. } 18'$$

where, as before, ω is the angular speed with which the body is spinning, while Ω is the angular speed with which the axis of spin is changing direction

This case is well illustrated by the ordinary gyroscope; also by the motion of the earth, a body in which the constant rate of spin ω is 2π radians a day, and in which the axis of spin has its direction changed at the rate Ω by the pull of the moon upon the equatorial belt of the earth. Observe that just as in Eq. 18 the three quantities a, S, and ω are vectors, each at right angles to the other two, so here in Eq. 18′, A, Ω, and ω are three mutually rectangular vectors.

The student will find it an interesting exercise to prove that angular and linear accelerations at any point on a rotating rigid body are connected by the following equation,

$$a = rA, \qquad \text{Eq. } 8'$$

an equation which is strictly analogous to Eq. 8, p. 28.

This subject is one whose detailed discussion at this point would lead us too far afield; but it is hoped that every student will at least get a fair grasp of the exact analogy between translation and rotation in Kinematics, and thus reduce the complexity of the subject by at least one half.

ALGEBRAIC SUMMARY OF KINEMATICS

Linear	Angular
Fundamental Quantities	
Length and time; p and t.	Angle and time; θ and t.