CALCULUS MADE EASY

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CALCULUS MADE EASY:

BEING A VERY-SIMPLEST INTRODUCTION TO THOSE BEAUTIFUL METHODS OF RECKONING WHICH ARE GENERALLY CALLED BY THE TERRIFYING NAMES OF THE

DIFFERENTIAL CALCULUS

AND THE

INTEGRAL CALCULUS.

BY

F. R. S.

SECOND EDITION, ENLARGED

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1914

What one fool can do, another can.

(*Ancient Simian Proverb.*)

PREFACE TO THE SECOND EDITION.

The surprising success of this work has led the author to add a considerable number of worked examples and exercises. Advantage has also been taken to enlarge certain parts where experience showed that further explanations would be useful.

The author acknowledges with gratitude many valuable suggestions and letters received from teachers, students, and—critics.

October, 1914.

CONTENTS.

PROLOGUE.

CONSIDERING how many fools can calculate, it is surprising that it should be thought either a difficult or a tedious task for any other fool to learn how to master the same tricks.

Some calculus-tricks are quite easy. Some are enormously difficult. The fools who write the textbooks of advanced mathematics—and they are mostly clever fools—seldom take the trouble to show you how easy the easy calculations are. On the contrary, they seem to desire to impress you with their tremendous cleverness by going about it in the most difficult way.

Being myself a remarkably stupid fellow, I have had to unteach myself the difficulties, and now beg to present to my fellow fools the parts that are not hard. Master these thoroughly, and the rest will follow. What one fool can do, another can.

CHAPTER I.

TO DELIVER YOU FROM THE PRELIMINARY TERRORS.

The preliminary terror, which chokes off most fifth-form boys from even attempting to learn how to calculate, can be abolished once for all by simply stating what is the meaning—in common-sense terms—of the two principal symbols that are used in calculating.

These dreadful symbols are:

(1) d which merely means "a little bit of."

Thus dx means a little bit of x; or du means a little bit of u. Ordinary mathematicians think it more polite to say "an element of," instead of "a little bit of." Just as you please. But you will find that these little bits (or elements) may be considered to be indefinitely small.

(2) $\int$ which is merely a long S, and may be called (if you like) "the sum of."

Thus $\int dx$ means the sum of all the little bits of x; or $\int dt$ means the sum of all the little bits of t. Ordinary mathematicians call this symbol "the integral of." Now any fool can see that if x is considered as made up of a lot of little bits, each of which is called dx, if you add them all up together you get the sum of all the dx's, (which is the same thing as the whole of x). The word "integral" simply means "the whole." If you think of the duration of time for one hour, you may (if you like) think of it as cut up into 3600 little bits called seconds. The whole of the 3600 little bits added up together make one hour.

When you see an expression that begins with this terrifying symbol, you will henceforth know that it is put there merely to give you instructions that you are now to perform the operation (if you can) of totalling up all the little bits that are indicated by the symbols that follow.

That's all.

CHAPTER II.

ON DIFFERENT DEGREES OF SMALLNESS.

We shall find that in our processes of calculation we have to deal with small quantities of various degrees of smallness.

We shall have also to learn under what circumstances we may consider small quantities to be so minute that we may omit them from consideration. Everything depends upon relative minuteness.

Before we fix any rules let us think of some familiar cases. There are 60 minutes in the hour, 24 hours in the day, 7 days in the week. There are therefore 1440 minutes in the day and 10080 minutes in the week.

Obviously 1 minute is a very small quantity of time compared with a whole week. Indeed, our forefathers considered it small as compared with an hour, and called it "one minùte," meaning a minute fraction—namely one sixtieth—of an hour. When they came to require still smaller subdivisions of time, they divided each minute into 60 still smaller parts, which, in Queen Elizabeth's days, they called "second minùtes" (*i.e.* small quantities of the second order of minuteness). Nowadays we call these small quantities of the second order of smallness "seconds." But few people know *why* they are so called.

Now if one minute is so small as compared with a whole day, how much smaller by comparison is one second!

Again, think of a farthing as compared with a sovereign: it is barely worth more than $\frac{1}{1000}$ part. A farthing more or less is of precious little importance compared with a sovereign: it may certainly be regarded as a *small* quantity. But compare a farthing with £1000: relatively to this greater sum, the farthing is of no more importance than $\frac{1}{1000}$ of a farthing would be to a sovereign. Even a golden sovereign is relatively a negligible quantity in the wealth of a millionaire.

Now if we fix upon any numerical fraction as constituting the proportion which for any purpose we call relatively small, we can easily state other fractions of a higher degree of smallness. Thus if, for the purpose of time, $\frac{1}{60}$ be called a *small* fraction, then $\frac{1}{60}$ of $\frac{1}{60}$ (being a *small* fraction of a *small* fraction) may be regarded as a *small quantity of the second order* of smallness.*

Or, if for any purpose we were to take 1 per cent. (*i.e.* $\frac{1}{100}$) as a *small* fraction, then 1 per cent. of 1 per cent. (*i.e.* $\frac{1}{10,000}$) would be a small fraction of the second order of smallness; and $\frac{1}{1,000,000}$ would be a small fraction of the third order of smallness, being 1 per cent. of 1 per cent. of 1 per cent.

Lastly, suppose that for some very precise purpose we should regard $\frac{1}{1,000,000}$ as "small." Thus, if a first-rate chronometer is not to lose or gain more than half a minute in a year, it must keep time with an accuracy of 1 part in $1,051,200$. Now if, for such a purpose, we

*The mathematicians talk about the second order of "magnitude" (*i.e.* greatness) when they really mean second order of *smallness*. This is very confusing to beginners.

regard $\frac{1}{1{,}000{,}000}$ (or one millionth) as a small quantity, then $\frac{1}{1{,}000{,}000}$ of $\frac{1}{1{,}000{,}000}$, that is $\frac{1}{1{,}000{,}000{,}000{,}000}$ (or one billionth) will be a small quantity of the second order of smallness, and may be utterly disregarded, by comparison.

Then we see that the smaller a small quantity itself is, the more negligible does the corresponding small quantity of the second order become. Hence we know that *in all cases we are justified in neglecting the small quantities of the second—or third* (or higher)*—orders*, if only we take the small quantity of the first order small enough in itself.

But, it must be remembered, that small quantities if they occur in our expressions as factors multiplied by some other factor, may become important if the other factor is itself large. Even a farthing becomes important if only it is multiplied by a few hundred.

Now in the calculus we write dx for a little bit of x. These things such as dx, and du, and dy, are called "differentials," the differential of x, or of u, or of y, as the case may be. [You *read* them as *dee-eks*, or *dee-you*, or *dee-wy*.] If dx be a small bit of x, and relatively small of itself, it does not follow that such quantities as $x \cdot dx$, or $x^2\,dx$, or $a^x\,dx$ are negligible. But $dx \times dx$ would be negligible, being a small quantity of the second order.

A very simple example will serve as illustration.

Let us think of x as a quantity that can grow by a small amount so as to become $x+dx$, where dx is the small increment added by growth. The square of this is $x^2 + 2x \cdot dx + (dx)^2$. The second term is not negligible because it is a first-order quantity; while the third term is of the second order of smallness, being a bit of, a bit of x^2. Thus if we took dx to mean numerically, say, $\frac{1}{60}$ of x, then the second term would be $\frac{2}{60}$ of x^2, whereas the third term would be $\frac{1}{3600}$ of x^2. This last term is clearly less important than the second. But if we go further and take dx to mean only $\frac{1}{1000}$ of x, then the second term will be $\frac{2}{1000}$ of x^2, while the third term will be only $\frac{1}{1,000,000}$ of x^2.

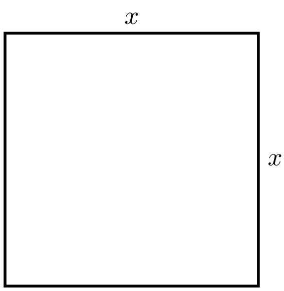

Fig. 1.

Geometrically this may be depicted as follows: Draw a square (Fig. 1) the side of which we will take to represent x. Now suppose the square to grow by having a bit dx added to its size each way. The enlarged square is made up of the original square x^2, the two rectangles at the top and on the right, each of which is of area $x \cdot dx$ (or together $2x \cdot dx$), and the little square at the top right-hand corner which is $(dx)^2$. In Fig. 2 we have taken dx as quite a big fraction of x—about $\frac{1}{5}$. But suppose we had taken it only $\frac{1}{100}$—about the thickness of an inked line drawn with a fine pen. Then the little corner square will have an area of only $\frac{1}{10,000}$ of x^2, and be practically invisible. Clearly $(dx)^2$ is negligible if only we consider the increment dx to be itself small enough.

Let us consider a simile.

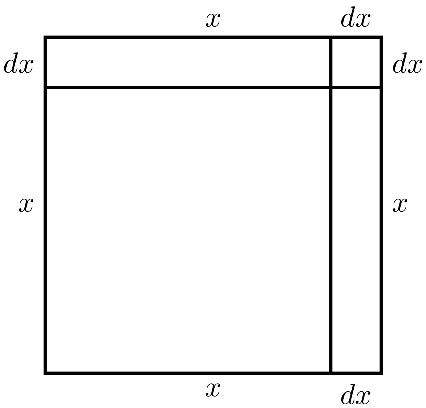

FIG. 2.

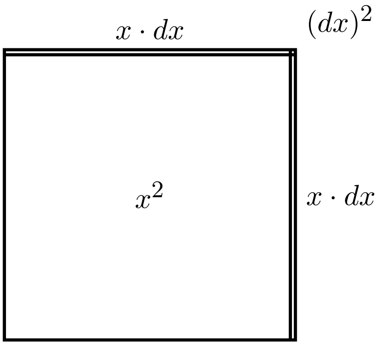

FIG. 3.

Suppose a millionaire were to say to his secretary: next week I will give you a small fraction of any money that comes in to me. Suppose that the secretary were to say to his boy: I will give you a small fraction of what I get. Suppose the fraction in each case to be $\frac{1}{100}$ part. Now if Mr. Millionaire received during the next week £1000, the secretary would receive £10 and the boy 2 shillings. Ten pounds would be a small quantity compared with £1000; but two shillings is a small small quantity indeed, of a very secondary order. But what would be the disproportion if the fraction, instead of being $\frac{1}{100}$, had been settled at $\frac{1}{1000}$ part? Then, while Mr. Millionaire got his £1000, Mr. Secretary would get only £1, and the boy less than one farthing!

The witty Dean Swift* once wrote:

> "So, Nat'ralists observe, a Flea
> "Hath smaller Fleas that on him prey.
> "And these have smaller Fleas to bite 'em,
> "And so proceed *ad infinitum*."

* *On Poetry: a Rhapsody* (p. 20), printed 1733—usually misquoted.

An ox might worry about a flea of ordinary size—a small creature of the first order of smallness. But he would probably not trouble himself about a flea's flea; being of the second order of smallness, it would be negligible. Even a gross of fleas' fleas would not be of much account to the ox.

CHAPTER III.

ON RELATIVE GROWINGS.

All through the calculus we are dealing with quantities that are growing, and with rates of growth. We classify all quantities into two classes: *constants* and *variables*. Those which we regard as of fixed value, and call *constants*, we generally denote algebraically by letters from the beginning of the alphabet, such as a, b, or c; while those which we consider as capable of growing, or (as mathematicians say) of "varying," we denote by letters from the end of the alphabet, such as x, y, z, u, v, w, or sometimes t.

Moreover, we are usually dealing with more than one variable at once, and thinking of the way in which one variable depends on the other: for instance, we think of the way in which the height reached by a projectile depends on the time of attaining that height. Or we are asked to consider a rectangle of given area, and to enquire how any increase in the length of it will compel a corresponding decrease in the breadth of it. Or we think of the way in which any variation in the slope of a ladder will cause the height that it reaches, to vary.

Suppose we have got two such variables that depend one on the other. An alteration in one will bring about an alteration in the other, *because* of this dependence. Let us call one of the variables x, and the other that depends on it y.

Suppose we make x to vary, that is to say, we either alter it or imagine it to be altered, by adding to it a bit which we call dx. We are thus causing x to become $x + dx$. Then, because x has been altered, y will have altered also, and will have become $y + dy$. Here the bit dy may be in some cases positive, in others negative; and it won't (except by a miracle) be the same size as dx.

Take two examples.

(1) Let x and y be respectively the base and the height of a right-angled triangle (Fig. 4), of which the slope of the other side is fixed

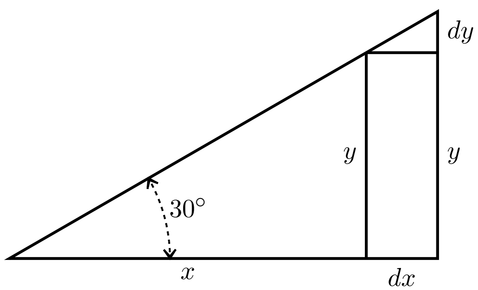

Fig. 4.

at 30°. If we suppose this triangle to expand and yet keep its angles the same as at first, then, when the base grows so as to become $x+dx$, the height becomes $y + dy$. Here, increasing x results in an increase of y. The little triangle, the height of which is dy, and the base of which is dx, is similar to the original triangle; and it is obvious that the value of the ratio $\dfrac{dy}{dx}$ is the same as that of the ratio $\dfrac{y}{x}$. As the angle is 30° it will be seen that here

$$\frac{dy}{dx} = \frac{1}{1.73}.$$

(2) Let x represent, in Fig. 5, the horizontal distance, from a wall, of the bottom end of a ladder, AB, of fixed length; and let y be the

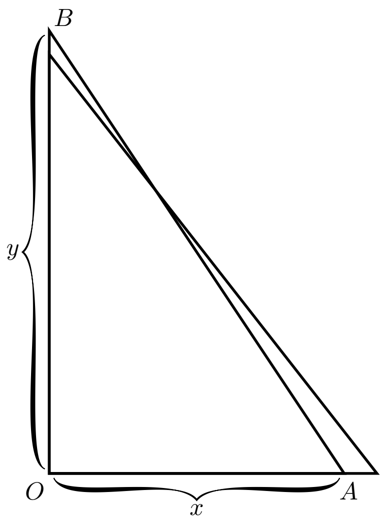

Fig. 5.

height it reaches up the wall. Now y clearly depends on x. It is easy to see that, if we pull the bottom end A a bit further from the wall, the top end B will come down a little lower. Let us state this in scientific language. If we increase x to $x+dx$, then y will become $y-dy$; that is, when x receives a positive increment, the increment which results to y is negative.

Yes, but how much? Suppose the ladder was so long that when the bottom end A was 19 inches from the wall the top end B reached just 15 feet from the ground. Now, if you were to pull the bottom end out 1 inch more, how much would the top end come down? Put it all into inches: $x = 19$ inches, $y = 180$ inches. Now the increment of x which we call dx, is 1 inch: or $x + dx = 20$ inches.

How much will y be diminished? The new height will be $y - dy$. If we work out the height by Euclid I. 47, then we shall be able to find how much dy will be. The length of the ladder is

$$\sqrt{(180)^2 + (19)^2} = 181 \text{ inches.}$$

Clearly then, the new height, which is $y - dy$, will be such that

$$(y - dy)^2 = (181)^2 - (20)^2 = 32761 - 400 = 32361,$$
$$y - dy = \sqrt{32361} = 179.89 \text{ inches.}$$

Now y is 180, so that dy is $180 - 179.89 = 0.11$ inch.

So we see that making dx an increase of 1 inch has resulted in making dy a decrease of 0.11 inch.

And the ratio of dy to dx may be stated thus:

$$\frac{dy}{dx} = -\frac{0.11}{1}.$$

It is also easy to see that (except in one particular position) dy will be of a different size from dx.

Now right through the differential calculus we are hunting, hunting, hunting for a curious thing, a mere ratio, namely, the proportion which dy bears to dx when both of them are indefinitely small.

It should be noted here that we can only find this ratio $\dfrac{dy}{dx}$ when y and x are related to each other in some way, so that whenever x varies y does vary also. For instance, in the first example just taken, if the base x of the triangle be made longer, the height y of the triangle becomes greater also, and in the second example, if the distance x of the foot of the ladder from the wall be made to increase, the height y

reached by the ladder decreases in a corresponding manner, slowly at first, but more and more rapidly as x becomes greater. In these cases the relation between x and y is perfectly definite, it can be expressed mathematically, being $\dfrac{y}{x} = \tan 30^\circ$ and $x^2 + y^2 = l^2$ (where l is the length of the ladder) respectively, and $\dfrac{dy}{dx}$ has the meaning we found in each case.

If, while x is, as before, the distance of the foot of the ladder from the wall, y is, instead of the height reached, the horizontal length of the wall, or the number of bricks in it, or the number of years since it was built, any change in x would naturally cause no change whatever in y; in this case $\dfrac{dy}{dx}$ has no meaning whatever, and it is not possible to find an expression for it. Whenever we use differentials dx, dy, dz, etc., the existence of some kind of relation between x, y, z, etc., is implied, and this relation is called a "function" in x, y, z, etc.; the two expressions given above, for instance, namely $\dfrac{y}{x} = \tan 30^\circ$ and $x^2+y^2 = l^2$, are functions of x and y. Such expressions contain implicitly (that is, contain without distinctly showing it) the means of expressing either x in terms of y or y in terms of x, and for this reason they are called *implicit functions* in x and y; they can be respectively put into the forms

$$y = x \tan 30^\circ \quad \text{or} \quad x = \frac{y}{\tan 30^\circ}$$

and

$$y = \sqrt{l^2 - x^2} \quad \text{or} \quad x = \sqrt{l^2 - y^2}.$$

These last expressions state explicitly (that is, distinctly) the value of x in terms of y, or of y in terms of x, and they are for this reason called *explicit functions* of x or y. For example $x^2 + 3 = 2y - 7$ is an implicit function in x and y; it may be written $y = \dfrac{x^2 + 10}{2}$ (explicit function of x) or $x = \sqrt{2y - 10}$ (explicit function of y). We see that an explicit function in x, y, z, etc., is simply something the value of which changes when x, y, z, etc., are changing, either one at the time or several together. Because of this, the value of the explicit function is called the *dependent variable*, as it depends on the value of the other variable quantities in the function; these other variables are called the *independent variables* because their value is not determined from the value assumed by the function. For example, if $u = x^2 \sin\theta$, x and θ are the independent variables, and u is the dependent variable.

Sometimes the exact relation between several quantities x, y, z either is not known or it is not convenient to state it; it is only known, or convenient to state, that there is some sort of relation between these variables, so that one cannot alter either x or y or z singly without affecting the other quantities; the existence of a function in x, y, z is then indicated by the notation $F(x, y, z)$ (implicit function) or by $x = F(y, z)$, $y = F(x, z)$ or $z = F(x, y)$ (explicit function). Sometimes the letter f or ϕ is used instead of F, so that $y = F(x)$, $y = f(x)$ and $y = \phi(x)$ all mean the same thing, namely, that the value of y depends on the value of x in some way which is not stated.

We call the ratio $\dfrac{dy}{dx}$ "the *differential coefficient* of y with respect to x." It is a solemn scientific name for this very simple thing. But we are not going to be frightened by solemn names, when the things themselves are so easy. Instead of being frightened we will simply pronounce a brief curse on the stupidity of giving long crack-jaw names; and, having relieved our minds, will go on to the simple thing itself, namely the ratio $\frac{dy}{dx}$.

In ordinary algebra which you learned at school, you were always hunting after some unknown quantity which you called x or y; or sometimes there were two unknown quantities to be hunted for simultaneously. You have now to learn to go hunting in a new way; the fox being now neither x nor y. Instead of this you have to hunt for this curious cub called $\frac{dy}{dx}$. The process of finding the value of $\frac{dy}{dx}$ is called "differentiating." But, remember, what is wanted is the value of this ratio when both dy and dx are themselves indefinitely small. The true value of the differential coefficient is that to which it approximates in the limiting case when each of them is considered as infinitesimally minute.

Let us now learn how to go in quest of $\frac{dy}{dx}$.

NOTE TO CHAPTER III.

How to read Differentials.

It will never do to fall into the schoolboy error of thinking that dx means d times x, for d is not a factor—it means "an element of" or "a bit of" whatever follows. One reads dx thus: "dee-eks."

In case the reader has no one to guide him in such matters it may here be simply said that one reads differential coefficients in the following way. The differential coefficient

$\dfrac{dy}{dx}$ is read "*dee-wy by dee-eks,*" or "*dee-wy over dee-eks.*"

So also $\dfrac{du}{dt}$ is read "*dee-you by dee-tee.*"

Second differential coefficients will be met with later on. They are like this:

$\dfrac{d^2y}{dx^2}$; which is read "*dee-two-wy over dee-eks-squared,*"

and it means that the operation of differentiating y with respect to x has been (or has to be) performed twice over.

Another way of indicating that a function has been differentiated is by putting an accent to the symbol of the function. Thus if $y = F(x)$, which means that y is some unspecified function of x (see p. 13), we may write $F'(x)$ instead of $\dfrac{d\big(F(x)\big)}{dx}$. Similarly, $F''(x)$ will mean that the original function $F(x)$ has been differentiated twice over with respect to x.

CHAPTER IV.

SIMPLEST CASES.

NOW let us see how, on first principles, we can differentiate some simple algebraical expression.

Case 1.

Let us begin with the simple expression $y = x^2$. Now remember that the fundamental notion about the calculus is the idea of *growing*. Mathematicians call it *varying*. Now as y and x^2 are equal to one another, it is clear that if x grows, x^2 will also grow. And if x^2 grows, then y will also grow. What we have got to find out is the proportion between the growing of y and the growing of x. In other words our task is to find out the ratio between dy and dx, or, in brief, to find the value of $\frac{dy}{dx}$.

Let x, then, grow a little bit bigger and become $x + dx$; similarly, y will grow a bit bigger and will become $y+dy$. Then, clearly, it will still be true that the enlarged y will be equal to the square of the enlarged x. Writing this down, we have:

$$y + dy = (x + dx)^2.$$

Doing the squaring we get:

$$y + dy = x^2 + 2x \cdot dx + (dx)^2.$$

What does $(dx)^2$ mean? Remember that dx meant a bit—a little bit—of x. Then $(dx)^2$ will mean a little bit of a little bit of x; that is, as explained above (p. 4), it is a small quantity of the second order of smallness. It may therefore be discarded as quite inconsiderable in comparison with the other terms. Leaving it out, we then have:

$$y + dy = x^2 + 2x \cdot dx.$$

Now $y = x^2$; so let us subtract this from the equation and we have left

$$dy = 2x \cdot dx.$$

Dividing across by dx, we find

$$\frac{dy}{dx} = 2x.$$

Now *this** is what we set out to find. The ratio of the growing of y to the growing of x is, in the case before us, found to be $2x$.

* *N.B.*—This ratio $\frac{dy}{dx}$ is the result of differentiating y with respect to x. Differentiating means finding the differential coefficient. Suppose we had some other function of x, as, for example, $u = 7x^2 + 3$. Then if we were told to differentiate this with respect to x, we should have to find $\frac{du}{dx}$, or, what is the same thing, $\frac{d(7x^2+3)}{dx}$. On the other hand, we may have a case in which time was the independent variable (see p. 14), such as this: $y = b + \frac{1}{2}at^2$. Then, if we were told to differentiate it, that means we must find its differential coefficient with respect to t. So that then our business would be to try to find $\frac{dy}{dt}$, that is, to find $\frac{d(b+\frac{1}{2}at^2)}{dt}$.

Numerical example.

Suppose $x = 100$ and $\therefore y = 10,000$. Then let x grow till it becomes 101 (that is, let $dx = 1$). Then the enlarged y will be $101 \times 101 = 10,201$. But if we agree that we may ignore small quantities of the second order, 1 may be rejected as compared with $10,000$; so we may round off the enlarged y to $10,200$. y has grown from $10,000$ to $10,200$; the bit added on is dy, which is therefore 200.

$\frac{dy}{dx} = \frac{200}{1} = 200$. According to the algebra-working of the previous paragraph, we find $\frac{dy}{dx} = 2x$. And so it is; for $x = 100$ and $2x = 200$.

But, you will say, we neglected a whole unit.

Well, try again, making dx a still smaller bit.

Try $dx = \frac{1}{10}$. Then $x + dx = 100.1$, and

$$(x + dx)^2 = 100.1 \times 100.1 = 10,020.01.$$

Now the last figure 1 is only one-millionth part of the $10,000$, and is utterly negligible; so we may take $10,020$ without the little decimal at the end. And this makes $dy = 20$; and $\frac{dy}{dx} = \frac{20}{0.1} = 200$, which is still the same as $2x$.

Case 2.

Try differentiating $y = x^3$ in the same way.

We let y grow to $y + dy$, while x grows to $x + dx$.

Then we have

$$y + dy = (x + dx)^3.$$

Doing the cubing we obtain

$$y + dy = x^3 + 3x^2 \cdot dx + 3x(dx)^2 + (dx)^3.$$