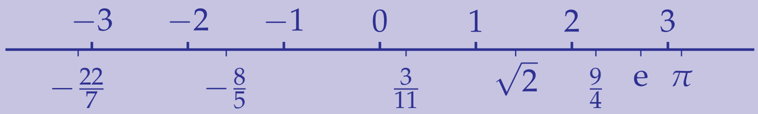
I Numbers



Part I is about the arithmetic of numbers. There are various types of numbers: positive numbers, negative numbers, integers, fractions, rational and irrational numbers. Examples of irrational numbers are $\sqrt{2}$, π and e. In higher mathematics, one also uses imaginary and complex numbers, but in this book we restrict ourselves to *real numbers*, i.e. numbers that can be represented as points on a number line.

The first two chapters are a recap of your skills in primary school arithmetic: addition, subtraction, multiplication, and division of integers and fractions. Chapter 3 treats the properties of powers and roots.

1 Calculating with integers

Perform the following calculations:

1.1

a.
$$\begin{array}{rl} 873 & \\ 112 & \\ 1718 & \\ 157 & \\ 3461 & \\ \hline & + \\ \ldots & \end{array}$$

b.
$$\begin{array}{rl} 1578 & \\ 9553 & \\ 7218 & \\ 212 & \\ 4139 & \\ \hline & + \\ \ldots & \end{array}$$

1.2

a.
$$\begin{array}{rl} 9134 & \\ 4319 & \\ \hline & - \\ \ldots & \end{array}$$

b.
$$\begin{array}{rl} 4585 & \\ 3287 & \\ \hline & - \\ \ldots & \end{array}$$

c.
$$\begin{array}{rl} 7033 & \\ 1398 & \\ \hline & - \\ \ldots & \end{array}$$

1.3 Calculate:

a. 34×89
b. 67×46
c. 61×93
d. 55×11
e. 78×38

1.4 Calculate:

a. 354×83
b. 67×546
c. 461×79
d. 655×102
e. 178×398

Find the quotient and the remainder by using a long division:

1.5

a. $154 : 13$
b. $435 : 27$
c. $631 : 23$
d. $467 : 17$
e. $780 : 37$

1.6

a. $2334 : 53$
b. $6463 : 101$
c. $7682 : 59$
d. $6178 : 451$
e. $5811 : 67$

1.7

a. $15457 : 11$
b. $4534 : 97$
c. $63321 : 23$
d. $56467 : 179$
e. $78620 : 307$

1.8

a. $42334 : 41$
b. $13467 : 101$
c. $35641 : 99$
d. $16155 : 215$
e. $92183 : 83$

Addition, subtraction and multiplication

The sequence

$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots$

enumerates the *positive integers*. Every child learns to count in this way. Addition, subtraction and multiplication with such numbers by hand are learned in primary school. Examples are given to the right.

$$
\begin{array}{rl}
341 & \\
295 & \\
718 & \\
12 & \\
1431 & + \\
\hline
2797 &
\end{array}
\qquad
\begin{array}{rl}
8135 & \\
3297 & - \\
\hline
4838 &
\end{array}
\qquad
\begin{array}{rl}
431 & \\
728 & \times \\
\hline
3448 & \\
862 & \\
3017 & \\
\hline
313768 &
\end{array}
$$

Long division

Division by hand is done by *long division*. To the right, the long division for 83218 : 37, i.e. 83218 divided by 37, is shown. The *quotient* 2249 is constructed digit by digit, from left to right, above the dividend. Each digit of the quotient is positioned exactly above the corresponding product in the 'tail' of the calculation. The *remainder* 5 is found below, at the end of the calculation. The long division shows that

$$
\begin{array}{r|l}
 & 2249 \quad \leftarrow \text{quotient} \\
\hline
37 & 83218 \\
 & 74 \\
 & \overline{92} \\
 & 74 \\
 & \overline{181} \\
 & 148 \\
 & \overline{338} \\
 & 333 \\
 & \overline{5} \quad \leftarrow \text{remainder}
\end{array}
$$

$$83218 = 2249 \times 37 + 5$$

This can also be written as

$$\frac{83218}{37} = 2249 + \frac{5}{37}$$

The right-hand side is usually simplified to $2249\frac{5}{37}$, which yields

$$\frac{83218}{37} = 2249\frac{5}{37}$$

Decompose the following numbers into prime factors:

1.9
- a. 24
- b. 72
- c. 250
- d. 96
- e. 98

1.10
- a. 288
- b. 1024
- c. 315
- d. 396
- e. 1875

1.11
- a. 972
- b. 676
- c. 2025
- d. 1122
- e. 860

1.12
- a. 255
- b. 441
- c. 722
- d. 432
- e. 985

1.13
- a. 2000
- b. 2001
- c. 2002
- d. 2003
- e. 2004

1.14
- a. your age in months
- b. your year of birth
- c. your PIN code

Find all divisors of the following numbers. Proceed accurately and systematically, since otherwise you risk missing a few. It is a good idea to write down the prime factorization of each number first.

1.15
- a. 12
- b. 20
- c. 32
- d. 108
- e. 144

1.16
- a. 72
- b. 100
- c. 1001
- d. 561
- e. 196

Divisors and prime numbers

Sometimes, a division yields a quotient without a remainder, i.e. with remainder 0. For instance, $238 : 17 = 14$. This means that $238 = 14 \times 17$. The numbers 14 and 17 are called *divisors* of 238 and the notation $238 = 14 \times 17$ is called a *decomposition into factors* or *factorization* of 238. The words 'divisor' and 'factor' are synonymous in this respect.

The divisor 14 itself can also be decomposed, namely as $14 = 2 \times 7$, but a further decomposition of 238 is not possible, since 2, 7 and 17 are *prime numbers*, i.e. numbers that cannot be decomposed into smaller factors. In this way, we have found the *prime factorization* $238 = 2 \times 7 \times 17$.

Since $238 = 1 \times 238$ is also a decomposition of 238, the numbers 1 en 238 are divisors of 238. Every number has 1 and itself as divisor. The interesting, *proper* divisors, however, are the divisors that are greater than 1 and smaller than the number itself. The prime numbers are the numbers greater than 1 that have no proper divisors. The sequence of prime numbers starts as follows:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, \ldots$$

Every integer greater than 1 can be decomposed into prime factors. To the right, examples are given of how to find such a *prime factorization* by systematically looking for bigger and bigger prime factors. Each time you find one, you divide by it and proceed with the quotient.

$$\begin{array}{rl} 180 & \\ \hline 90 & 2 \\ \hline 45 & 2 \\ \hline 15 & 3 \\ \hline 5 & 3 \\ \hline 1 & 5 \end{array} \qquad \begin{array}{rl} 585 & \\ \hline 195 & 3 \\ \hline 65 & 3 \\ \hline 13 & 5 \\ \hline 1 & 13 \end{array} \qquad \begin{array}{rl} 3003 & \\ \hline 1001 & 3 \\ \hline 143 & 7 \\ \hline 13 & 11 \\ \hline 1 & 13 \end{array}$$

You are done as soon as you end up with a quotient of 1. The prime factors are then collected on the right-hand side of the ladder. The three ladder diagrams yield the prime factorization:

$$\begin{array}{rcl@{\quad}c@{\quad}l} 180 & = & 2 \times 2 \times 3 \times 3 \times 5 & = & 2^2 \times 3^2 \times 5 \\ 585 & = & 3 \times 3 \times 5 \times 13 & = & 3^2 \times 5 \times 13 \\ 3003 & = & 3 \times 7 \times 11 \times 13 & & \end{array}$$

As shown, it is convenient to write prime factors that occur more than once in power notation: $2^2 = 2 \times 2$ and $3^2 = 3 \times 3$. Some more examples are given below (you may construct the ladder diagrams for yourself):

$$\begin{array}{rcl@{\quad}c@{\quad}l} 120 & = & 2 \times 2 \times 2 \times 3 \times 5 & = & 2^3 \times 3 \times 5 \\ 81 & = & 3 \times 3 \times 3 \times 3 & = & 3^4 \\ 48 & = & 2 \times 2 \times 2 \times 2 \times 3 & = & 2^4 \times 3 \end{array}$$

Find the greatest common divisor (gcd) of:

1.17

a. 12 and 30
b. 24 and 84
c. 27 and 45
d. 32 and 56
e. 34 and 85

1.18

a. 45 and 225
b. 144 and 216
c. 90 and 196
d. 243 and 135
e. 288 and 168

1.19

a. 1024 and 864
b. 1122 and 1815
c. 875 and 1125
d. 1960 and 6370
e. 1024 and 1152

1.20

a. 1243 and 1244
b. 1721 and 1726
c. 875 and 900
d. 1960 and 5880
e. 1024 and 2024

Find the least common multiple (lcm) of:

1.21

a. 12 and 30
b. 27 and 45
c. 18 and 63
d. 16 and 40
e. 33 and 121

1.22

a. 52 and 39
b. 64 and 80
c. 144 and 240
d. 169 and 130
e. 68 and 51

1.23

a. 250 and 125
b. 144 and 216
c. 520 and 390
d. 888 and 185
e. 124 and 341

1.24

a. 240 and 180
b. 276 and 414
c. 588 and 504
d. 315 and 189
e. 403 and 221

Find the gcd and the lcm of:

1.25

a. 9, 12 and 30
b. 24, 30 and 36
c. 10, 15 and 35
d. 18, 27 and 63
e. 21, 24 and 27

1.26

a. 28, 35 and 49
b. 64, 80 and 112
c. 39, 52 and 130
d. 144, 168 and 252
e. 189, 252 and 315

The gcd and the lcm

Two numbers can have common divisors. The *greatest common divisor* (gcd) is of particular interest. If the decomposition into prime factors of both numbers is known, their gcd can be found immediately. For instance, on page 9 we calculated the following prime factorizations:

$$\begin{aligned} 180 &= 2^2 \times 3^2 \times 5 \\ 585 &= 3^2 \times 5 \times 13 \\ 3003 &= 3 \times 7 \times 11 \times 13 \end{aligned}$$

From this we see that

$$\begin{aligned} \gcd(180, 585) &= \gcd(2^2 \times 3^2 \times 5\,,\, 3^2 \times 5 \times 13) = 3^2 \times 5 = 45 \\ \gcd(180, 3003) &= \gcd(2^2 \times 3^2 \times 5\,,\, 3 \times 7 \times 11 \times 13) = 3 \\ \gcd(585, 3003) &= \gcd(3^2 \times 5 \times 13\,,\, 3 \times 7 \times 11 \times 13) = 3 \times 13 = 39 \end{aligned}$$

The *least common multiple* (lcm) of two numbers is the smallest number that is a multiple of both numbers. In other words, it is the smallest number that may be divided without a remainder by both numbers. The lcm can also be found immediately, based on the prime factorizations of the numbers. For instance:

$$\operatorname{lcm}(180, 585) = \operatorname{lcm}(2^2 \times 3^2 \times 5\,,\, 3^2 \times 5 \times 13) = 2^2 \times 3^2 \times 5 \times 13 = 2340$$

A convenient property of the gcd and the lcm of two numbers is the fact that, when they are multiplied, their product equals the product of the two numbers. For instance:

$$\gcd(180, 585) \times \operatorname{lcm}(180, 585) = 45 \times 2340 = 105300 = 180 \times 585$$

The gcd and the lcm can also be calculated for more than two numbers, based on their prime factorizations. For instance:

$$\begin{aligned} \gcd(180, 585, 3003) &= 3 \\ \operatorname{lcm}(180, 585, 3003) &= 2^2 \times 3^2 \times 5 \times 7 \times 11 \times 13 = 180180 \end{aligned}$$

A smart idea

There is a method to calculate the gcd of two numbers without prime factorization. In many cases, this method is much faster. The basic idea is that the gcd of two numbers is also a divisor of their *difference*. (Do you know why?)
For instance, $\gcd(4352, 4342)$ must also be a divisor of $4352 - 4342 = 10$. The number 10 only has the prime factors 2 and 5. Obviously, 5 is not a common divisor of both numbers, but 2 is, so we have $\gcd(4352, 4342) = 2$. If you are smart, you may save yourself lots of time using this idea!

2 Calculating with fractions

2.1 Simplify:

a. $\frac{15}{20}$

b. $\frac{18}{45}$

c. $\frac{21}{49}$

d. $\frac{27}{81}$

e. $\frac{24}{96}$

2.2 Simplify:

a. $\frac{60}{144}$

b. $\frac{144}{216}$

c. $\frac{135}{243}$

d. $\frac{864}{1024}$

e. $\frac{168}{288}$

2.3 Write with common denominator:

a. $\frac{1}{3}$ and $\frac{1}{4}$

b. $\frac{2}{5}$ and $\frac{3}{7}$

c. $\frac{4}{9}$ and $\frac{2}{5}$

d. $\frac{7}{11}$ and $\frac{3}{4}$

e. $\frac{2}{13}$ and $\frac{5}{12}$

2.4 Write with common denominator:

a. $\frac{1}{6}$ and $\frac{1}{9}$

b. $\frac{3}{10}$ and $\frac{2}{15}$

c. $\frac{3}{8}$ and $\frac{5}{6}$

d. $\frac{5}{9}$ and $\frac{7}{12}$

e. $\frac{3}{20}$ and $\frac{1}{8}$

2.5 Write with common denominator:

a. $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{5}$

b. $\frac{2}{3}$, $\frac{3}{5}$ and $\frac{2}{7}$

c. $\frac{1}{4}$, $\frac{1}{6}$ and $\frac{1}{9}$

d. $\frac{2}{10}$, $\frac{1}{15}$ and $\frac{5}{6}$

e. $\frac{5}{12}$, $\frac{7}{18}$ and $\frac{3}{8}$

2.6 Write with common denominator:

a. $\frac{2}{27}$, $\frac{5}{36}$ and $\frac{5}{24}$

b. $\frac{7}{15}$, $\frac{3}{20}$ and $\frac{5}{6}$

c. $\frac{4}{21}$, $\frac{3}{14}$ and $\frac{7}{30}$

d. $\frac{4}{63}$, $\frac{5}{42}$ and $\frac{1}{56}$

e. $\frac{5}{78}$, $\frac{5}{39}$ and $\frac{3}{65}$

Determine which of the given fractions is the greatest by writing them with a common denominator:

2.7

a. $\frac{5}{18}$ and $\frac{6}{19}$

b. $\frac{7}{15}$ and $\frac{5}{12}$

c. $\frac{9}{20}$ and $\frac{11}{18}$

d. $\frac{11}{36}$ and $\frac{9}{32}$

e. $\frac{20}{63}$ and $\frac{25}{72}$

2.8

a. $\frac{4}{7}$ and $\frac{2}{3}$

b. $\frac{14}{85}$ and $\frac{7}{51}$

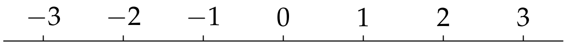
c. $\frac{26}{63}$ and $\frac{39}{84}$

d. $\frac{31}{90}$ and $\frac{23}{72}$

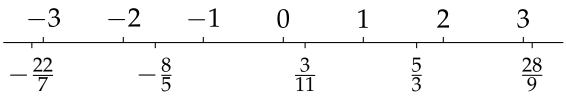
e. $\frac{37}{80}$ and $\frac{29}{60}$

Rational numbers

The sequence $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$ is the sequence of all integer numbers. The *number line* drawn below is a geometric representation of this sequence.



The *rational numbers*, i.e. the numbers that can be written as a fraction, also have their place on the number line. The number line below shows some rational numbers.



A fraction contains two integers, the *numerator* and the *denominator*, separated by a horizontal line or a slanting line. For example, in the fraction $\frac{28}{6}$ the numerator is 28 and the denominator is 6. The denominator of a fraction cannot be zero. A rational number is a number that can be written as a fraction, but this notation is not unique: if you multiply the numerator and denominator by the same integer (not equal to 0) or divide both by a common factor greater than 1, this does not change its place on the number line. For instance:

$$\frac{28}{6} = \frac{14}{3} = \frac{-14}{-3} = \frac{70}{15}$$

Fractions like $\frac{-5}{3}$ and $\frac{22}{-7}$ are usually written as $-\frac{5}{3}$ and $-\frac{22}{7}$, respectively. Integers can also be written as fractions, e.g. $7 = \frac{7}{1}$, $-3 = -\frac{3}{1}$ and $0 = \frac{0}{1}$. This shows that integers are rational numbers as well.

Dividing the numerator and the denominator of a fraction by the same factor is called *simplifying* the fraction. For instance, $\frac{28}{6}$ may be simplified to $\frac{14}{3}$ by dividing both the numerator and denominator by 2. A fraction is called *irreducible* or *in lowest terms* if the greatest common divisor (gcd) of the numerator and denominator equals 1. Thus, $\frac{14}{3}$ is in lowest terms, while $\frac{28}{6}$ is not. Every fraction can be written in lowest terms by dividing the numerator and denominator by their gcd.

Two fractions always can be rewritten with a common denominator. For instance, $\frac{4}{15}$ and $\frac{5}{21}$ can be written with a common denominator $15 \times 21 = 315$ since $\frac{4}{15} = \frac{84}{315}$ and $\frac{5}{21} = \frac{75}{315}$. But taking the least common multiple (lcm) of the denominators, in this case $\mathrm{lcm}(15, 21) = 105$, yields the simplest fractions with a common denominator, namely $\frac{28}{105}$ and $\frac{25}{105}$.

Calculate:

2.9

a. $\frac{1}{3}+\frac{1}{4}$

b. $\frac{1}{5}-\frac{1}{6}$

c. $\frac{1}{7}+\frac{1}{9}$

d. $\frac{1}{9}-\frac{1}{11}$

e. $\frac{1}{2}+\frac{1}{15}$

2.10

a. $\frac{2}{3}+\frac{3}{4}$

b. $\frac{3}{5}-\frac{4}{7}$

c. $\frac{2}{7}+\frac{3}{4}$

d. $\frac{4}{9}-\frac{3}{8}$

e. $\frac{5}{11}+\frac{4}{15}$

2.11

a. $\frac{1}{6}+\frac{1}{4}$

b. $\frac{1}{9}-\frac{2}{15}$

c. $\frac{3}{8}+\frac{1}{12}$

d. $\frac{1}{3}+\frac{5}{6}$

e. $\frac{4}{15}-\frac{3}{10}$

2.12

a. $\frac{2}{45}+\frac{1}{21}$

b. $\frac{5}{27}-\frac{1}{36}$

c. $\frac{5}{72}+\frac{7}{60}$

d. $\frac{3}{34}+\frac{1}{85}$

e. $\frac{7}{30}+\frac{8}{105}$

2.13

a. $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$

b. $\frac{1}{2}-\frac{1}{3}+\frac{1}{7}$

c. $\frac{1}{4}-\frac{1}{5}+\frac{1}{9}$

d. $\frac{1}{2}-\frac{1}{7}-\frac{1}{3}$

e. $\frac{1}{8}+\frac{1}{3}-\frac{1}{5}$

2.14

a. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$

b. $\frac{1}{3}+\frac{1}{6}+\frac{1}{4}$

c. $\frac{1}{12}+\frac{1}{8}-\frac{1}{2}$

d. $\frac{1}{9}-\frac{1}{12}+\frac{1}{18}$

e. $\frac{1}{10}-\frac{1}{15}+\frac{1}{6}$

2.15

a. $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}$

b. $\frac{1}{3}+\frac{1}{6}-\frac{1}{4}$

c. $\frac{1}{12}-\frac{1}{8}-\frac{1}{2}$

d. $\frac{1}{9}-\frac{1}{12}-\frac{1}{18}$

e. $\frac{1}{10}+\frac{1}{15}+\frac{1}{6}$

2.16

a. $\frac{1}{3}-\frac{1}{9}+\frac{1}{27}$

b. $\frac{1}{2}+\frac{1}{10}-\frac{2}{15}$

c. $\frac{1}{18}-\frac{7}{30}-\frac{3}{20}$

d. $\frac{3}{14}-\frac{1}{21}+\frac{5}{6}$

e. $\frac{2}{5}-\frac{3}{10}+\frac{4}{15}$

2.17

a. $\frac{2}{5}-\frac{1}{7}-\frac{1}{10}$

b. $\frac{3}{2}+\frac{2}{3}-\frac{5}{6}$

c. $\frac{8}{21}-\frac{2}{7}+\frac{3}{4}$

d. $\frac{2}{11}-\frac{5}{13}+\frac{1}{2}$

e. $\frac{4}{17}-\frac{3}{10}+\frac{2}{5}$