Chapter 1

Elephant ears, dolphin fins and the balance equation

1.1 Keeping the balance

It is easy to come up with a set of questions concerning daily life observations that require some degree of physics in answering. At first sight some may be only remotely connected to physics, like the question:

- *Why do (African) elephants have such enormous ears?*

Part of the answer can only be given if the balance between production of heat and the desired temperature of the animal is taken into consideration. All mammals have to maintain a body temperature of about 37°C. However, even in rest the animal will use energy which is partly converted into heat within its body. Obviously, to arrive at a steady state as far as its temperature is concerned, the animal will have to transport this heat to the environment as can be seen from a very elementary (steady state) heat balance:

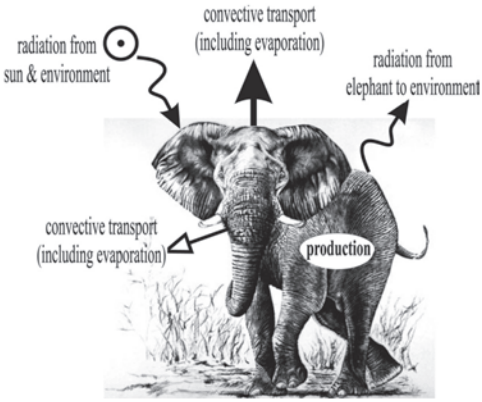

Figure 1.1 Heat production and flows from and to an elephant.

$$0 = \text{heat flow from environment - heat flow to environment} + \\ + \text{ internal heat production} \quad (1.1)$$

The heat production is proportional to the volume of the elephant.The heat flows are more complicated as there are various ways to transport heat from an object to its surroundings. Generally, we split these in three groups: heat transport by conduction, by convection and by radiation. The relative importance of these three depends on the particular circumstances. For instance, if the elephant is standing in the bright sunshine, it is obvious that he will receive radiation from the sun that will count as an inflow of heat. If there is wind, the elephant might be cooled by the wind if the air is colder than the outside of the elephant or heated if the air temperature is higher than its skin temperature.
Furthermore, for animals there is the very important possibility to loose heat due

to evaporation of water, i.e. by sweating. For all these flows of heat it holds that they are proportional to the surface of the elephant. So, if we look at the heat balance of the elephant, we see that the internal production of heat is proportional to the volume of the elephant and that the net balancing heat loss is to a good extend proportional to its surface. This gives us a good part of the explanation for the big ears: increase of (cooling) surface without increase of (heat producing) volume.

- *How does a dolphin in cold sea water prevent itself from too much heat losses via its flippers?*

This question can be investigated along the same lines as in the above example. Now we focus on the flippers. It will be clear that the blood that flows through them will be cooled substantially, being 'surrounded' by cold sea water. Nature has found a clever way of dealing with this, that humans (especially engineers) since the industrial revolution started to utilize frequently: the heat exchanger. In the flippers of the dolphin the veins carrying blood that flows from the body into the flippers are clearly twisted around the return veins that carry the blood back into the body (see Figure 1.2).

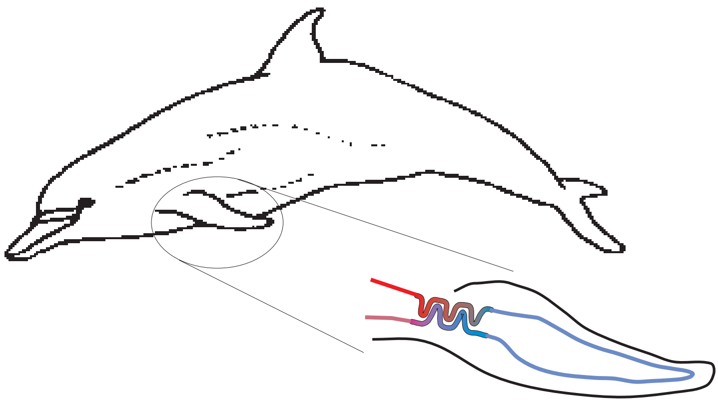

Figure 1.2 Schematic representation of the veins in the dolphin flippers.

What happens is that the warm blood coming from the dolphin body exchanges heat with the blood that returns to the body. The latter is obviously cooled by the cold surrounding sea. The driving force for the heat flow from the hot to the cold flow is the temperature difference between the two blood streams.

Intermezzo: balance equations

Figure 1.3 Changing number of people in a country due to in, out flow and birth and death.

In physics 'conserved quantities' form a special type. They are at the foundation of physical theories for the obvious reason that no matter what happens during an event a conserved quantity will come out the same as it went in. This does not mean that nothing is changing: the conserved quantity can be redistributed over the various parts participating in the event. In physics, mass, momentum and energy are amongst the most important conserved quantities. It is, however, in many cases more convenient to think of these quantities in terms of a balance rather than a conservation equation. The latter expresses that the quantity can not disappear. the former leaves room for 'production' of a quantity under consideration (where a negative production stands for destruction). An easy example, although not from physics, is the number of people in a country. Obviously, 'people' is not a conserved quantity: the total number of people varies over time, as there is birth and death. If these two were absent, then of course the total number of people would be conserved. For the country chosen, the number of people present at time $t+\Delta t$, let's call that $N(t+\Delta t)$ depends on the number at time t, $N(t)$ and on the number of people coming into the country during the time interval Δt, the number going out during Δt as well as the number of people that died and are born in this period. We group the latter two together into a net production, *Prod*. We can write this now in a balance equation:

$$N(t+\Delta t)-N(t)=in\,(\Delta t)-out\,(\Delta t)+Prod\,(\Delta t) \tag{1.2}$$

The terms '*in*' and '*out*' can be written more convenient as a mean flow in or out of the country (in number of people per second) acting during the time interval Δt. Note that this flow in and out requires actual crossing of the border of the country by the people. Similarly, we will write the production term as the number of people born in the country minus those who die in the country both per unit of time, multiplied by the time interval. Using these we can write the above equation as:

$$N(t+\Delta t)-N(t)=flow_{in}\cdot\Delta t-flow_{out}\cdot\Delta t+P\cdot\Delta t \tag{1.3}$$

By dividing both sides by Δt and taking the limit $\Delta t\to 0$, we arrive at the basic form of a balance equation, which describes the dynamics of the quantity under consideration, in the above example the number of people in the country.

$$\frac{dN}{dt}=flow_{in}-flow_{out}+P \tag{1.4}$$

In case of a steady state, obviously, the rate of change with time is zero and we get:

$$\text{st.st.}\quad\rightarrow\quad 0=flow_{in}-flow_{out}+P \tag{1.5}$$

Hence, this is a passive system that does not cost the dolphin any extra energy and furthermore is sensitive to the heat loss of the blood by its nature. The colder the returning blood the more heat is transferred from the hot blood to the cold. Consequently, the colder the hot blood becomes before being 'exposed' to the cold sea. In this way the system is self regulating and the temperature of the blood entering the dolphin's body is as high as possible without the input of extra energy by the animal.
In many technical applications heat exchangers (or mass exchangers for that matter) can be found that use a similar arrangement.

Chapter 2

The sky has a limit

2.1 Thickness of the atmosphere

The earth is a special planet. It has the right conditions to make life possible. It has water in enormous quantities and is blessed with an atmosphere containing oxygen and other gases. It has the right distance from its star, our sun, to allow for a temperature that leaves water liquid.

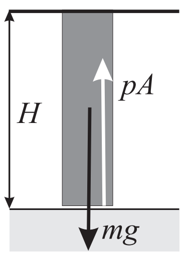

Figure 2.1 The pressure 'carrying' the atmosphere.

The earth radius is about 6378 km. The atmosphere forms only a thin layer around the earth. How thick is this layer? Let's make a guess. We do know that the pressure at the earth surface is 1 atm. Just like in water, this pressure can be explained as hydrostatic pressure: the total weight of the atmosphere is 'lifted' by the pressure. So, a first rough guess could be: assume the density of the air is constant (1.2 kg/m^3). Then, a simple force balance over a vertical column with cross-sectional area A of the atmosphere gives:

hydrostatic pressure

$$p * A - F_{grav} = 0 \Longrightarrow p * A - \rho A H g = 0 \Longrightarrow H = \frac{p}{\rho g} = 8.6\,\text{km} \tag{2.1}$$

Intermezzo: force balancing

A force balance can be set up for any material entity that is either at rest or moves at a constant velocity. This notion goes back to Newton, who formulated this as one of his main physical laws: given a mass at rest or moving at a constant velocity, then the sum of forces acting on this mass must be zero. In vector notation we can write this as:

$$\sum_i \vec{F}_i = 0 \tag{2.2}$$

This equation is a vector relation. Hence, it holds for the different components of the vector, *e.g.* for the x, y or z direction in a Cartesian coordinate system.

Obviously, this answer must be wrong as air planes routinely cruise at a height of 10 km. Thus, we need to improve our model. And it goes without saying that the constant density is a very dubious assumption. Of course, going up in the atmosphere, the density decreases. We can use the ideal gas law to relate the density to the pressure. The latter is also decreasing with height as the pressure at height z does not have to carry what is below z. But the density is also a function of the temperature. Let's try isothermal conditions, hence we set $T = 0°C$ =273 K.

Intermezzo: Ideal gas law

An ideal gas is a physical concept that describes the relation between the pressure, temperature and volume of a given number of moles of gas. Ideal gases do not really exist, but gases in which the molecular attraction between the molecules plays a negligible role are accurately described by it. Boyle and Gay-Lussac formulated the basic relations, that were later put together to form the ideal gas law:

$$pV = nRT \tag{2.3}$$

with p the pressure, V the volume occupied by the gas, n the number of moles of the gas and T the temperature. The constant R is the gas constant and has a value of 8.3144 J/molK.
Usually, dilute gases follow the ideal gas law. At temperatures well above the boiling point of nitrogen and oxygen, air can be treated as an ideal gas. By multiplying the above equation by the molar mass M, the ideal gas law describes the relation between pressure, temperature and density (ρ):

$$pV = nRT \rightarrow pM = \frac{nM}{V}RT = \rho RT \tag{2.4}$$

force balance: slice

Now we need to set up a force balance that takes into account the variation of the density with height. Therefore, consider a small slice out of a vertical column of the atmosphere, between z and $z+\Delta z$ (see Figure 2.2).

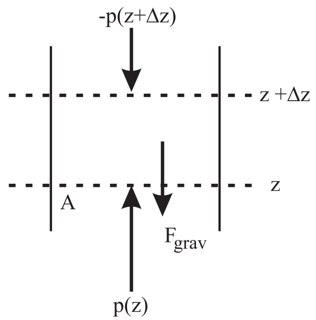

Figure 2.2 Forces on a small slice of the atmosphere.

The weight of this slice is: $\rho A \Delta z g$. Three forces act on this slice: at the bottom the pressure at that position pushes upwards: $p(z)A$; at the top the pressure pushed downwards: $p(z+\Delta z)A$ and, of course, gravity. Thus for a steady state we have according to Newton that the sum of the forces is zero:

$$p(z)A - p(z+\Delta z)A - \rho A \Delta z g = 0 \tag{2.5}$$

If we use a Taylor expansion on $p(z+\Delta z)$

$$p(z+\Delta z) = p(z) + \frac{dp}{dz}\Delta z + h.o.t. \tag{2.6}$$

we can simplify eq.(2.5) to

$$\frac{dp}{dz} = -\rho g \tag{2.7}$$

Intermezzo: Robert Boyle & Joseph Gay-Lussac

The ideal gas law is attached to the work of Robert Boyle (1627-1691) and Joseph Gay-Lussac (1778-1850). Boyle experimented with gases and reached an important conclusion, now known as Boyle's law: "For a gas under constant temperature, the volume is inversely proportional to pressure". In formula: $pV = const.$ at constant T. He stated that gases are made of tiny particles spaced very far apart. After improving Guericke's pump, he demonstrated that a feather and a lump of lead fall at the same speed in a vacuum.

Figure 2.3 Robert Boyle (1627-1691).

Figure 2.4 Joseph Gay-Lussac (1778-1850).

Joseph Louis Gay-Lussac extended the law of Boyle, by realising that all gases expand by equal amounts when subject to equal increments in temperature, if the pressure is kept constant: $V \propto T$ for constant p. By combining these two laws with the notion that gases are made of particles, the ideal law can be understood.

isothermal atmosphere

For isothermal conditions the density and pressure are linked according to:

$$pV = nRT \rightarrow p = \rho \frac{RT}{M} \tag{2.8}$$

with M the molar mass of air (= 28.8 $\cdot 10^{-3}$ kg/m^3). Combination of the last two equations gives:

$$\frac{1}{\rho}\frac{d\rho}{dz} = -\frac{gM}{RT} \tag{2.9}$$

Thus, for the density profile in the atmosphere (taking $z = 0 \rightarrow \rho = \rho_0$) is:

$$\frac{\rho(z)}{\rho_0} = exp\left(-\frac{gM}{RT}z\right) \tag{2.10}$$

The characteristic length scale is therefore: $\Lambda = \frac{RT}{gM} = 8.0km$. If we define rather arbitrarily the end of the atmosphere as the position where the density has dropped to $\rho_0/1000$, we find that the atmosphere has a thickness of 55km.

adiabatic atmo-sphere

Of course, we know that the atmosphere is not isothermal. Instead, we could try and assume that the atmosphere behaves 'adiabatic'. This means that if a portion of air moves up or down, it can adjust itself to the new surrounding without exchanging any heat. Of course, the ideal gas law is still valid, but the temperature is now no longer a constant with respect to height. Instead, it changes in such a way that pV^γ is constant (with $\gamma = \frac{c_p}{c_V} = 1.4$ for O_2 and N_2). If we use this relation in eq.(2.7), we find:

$$\rho^{\gamma-2}\frac{d\rho}{dz} = -\frac{\rho_0^\gamma g}{\gamma p_0} \quad (2.11)$$

with ρ_0 and p_0 the atmospheric density and pressure at ground level. The solution of the above equation reads as:

$$\rho(z) = \left(\rho_0^{\gamma-1} - \frac{\gamma-1}{\gamma}\frac{\rho_0^\gamma g}{p_0}z\right)^{\frac{1}{\gamma-1}} \quad (2.12)$$

If we use the last equation to find $z = H$ at which $\rho = 0$, we get $H = 30$ km.
A good approximation for the actual thickness is some 50km. There, the pressure has dropped to about 1Pa. In the figures below, the distribution of the pressure (calculated by integrating eq.(2.7) for both the isothermal and adiabatic solution) is given for the isothermal and adiabatic model of our atmosphere.

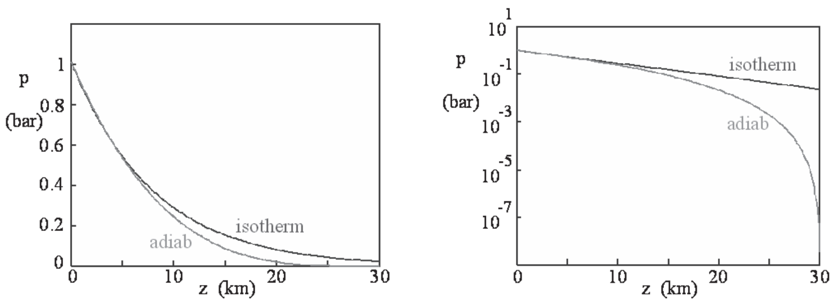

Figure 2.5 Pressure distribution of the atmosphere: (a) linear scale, (b) log-scale.

Intermezzo: Adiabatic processes

In an adiabatic process no heat is transfered from one part of the system to another. This can for instance be the case if the process is so fast that the heat flow simply has not enough time to transport significant amounts of heat.
The first law of Thermodynamics relates changes of heat to work done and to an increase of internal energy during the process. Actually, it is just another way of expressing that energy is a conserved quantity that can neither be lost nor created in a process. The first law of Thermodynamics is usually written as

$$dU = dQ - pdV \tag{2.13}$$

with dQ the amount of heat added to the system, dU the increase of internal energy of the system and pdV the amount of work performed on the surroundings by the system.
In case of an adiabatic process we have

$$\text{adiabatic} \rightarrow dQ = 0 \Rightarrow dU = -pdV \tag{2.14}$$

For an ideal gas, the internal energy is only a function of the temperature of the gas. This makes sense, as for an ideal gas the intermolecular forces are negligible and the only form of internal energy of the gas is the kinetic energy of the molecules. Temperature essentially is but a measure of the kinetic energy of these molecules. The internal energy and temperature are for an ideal gas related as:

$$U = \frac{5}{2}nRT \tag{2.15}$$

If we now combine the above equation with the adiabatic version of the first law of Thermodynamics, we obtain a relation between temperature, volume and pressure:

$$dU = pdV \rightarrow \frac{5}{2}d(nRT) = -pdV \tag{2.16}$$

Again, combining this with the ideal gas law, we can replace the term nRT by pV:

$$\frac{5}{2}dpV = -pdV \rightarrow \frac{7}{2}pdV = -\frac{5}{2}Vdp \rightarrow \frac{\frac{7}{2}}{\frac{5}{2}}\frac{dV}{V} = -\frac{dp}{p} \tag{2.17}$$

If we denote $\gamma \equiv \frac{7/2}{5/2}$, the above equation can be integrated to $pV^{\gamma} = const.$

2.1.1 Cooler mountains, lower boiling temperature

As a consequence of the above, areas at higher altitude are generally colder than those at sea-level.
From our adiabatic model, we can easily calculate the temperature change with height using the distribution of the density and pressure that we have derived. To obtain the temperature distribution we have to put these in the ideal gas law,

$pM = \rho RT$. Luckily, the adiabatic density distribution is a kind of power law: $(a-z)^p$. Integrating that, we find a pressure distribution of the form $(a-z)^{p+1}$. Consequently, the temperature distribution which is the ratio of p over ρ will thus be of the form $(a-z)^{p+1}/(a-z)^p = a-z$, thus linear in z! The precise answer is:

$$T(z) = \frac{M}{R}\left[\frac{p_0}{\rho_0} - \frac{\gamma-1}{\gamma} gz\right] \tag{2.18}$$

We can insert numbers: $M = 28.8 \cdot 10^{-3}$ kg/mol, $R = 8.3144$ J/molK, $p_0 = 101325$ Pa, $\rho_0 = 1.2$ kg/m^3, $T_0 = 20$°C, $g = 9.81$ m/s^2 and $\gamma = 1.4$.

temperature distribution

This gives for the temperature decrease (in SI-units):

$$T(z) = 20°C - 9.7 \cdot 10^{-3}\left(\frac{°C}{m}\right) \cdot z \tag{2.19}$$

which gives as a rule of thumb 1 degree for every 100 meters. Note, that the 20°C is just an example temperature. It varies with the time of the day, from day to day and from place to place. The temperature decrease with height, however, is universal as it depends only on M, R, g, γ, which are constants.

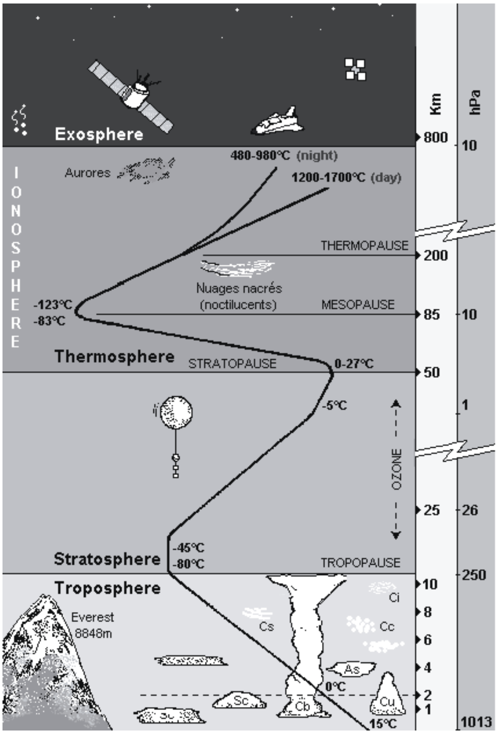

Figure 2.6 Temperature and pressure distribution in the atmosphere of the earth.

In reality, the temperature distribution is much more complicated. In our modeling above, we completely ignored the possibility of temperature variations due to thermals, pressure distributions due to variation in heat received from the sun over the globe, in other words the complicated dynamics of our atmosphere. A better impression is found in Figure 2.6. It shows an almost linear decrease with height for the lower part of the atmosphere, roughly for the first 10 km which is the troposphere in which we live. The slope of this decrease coincides with our predicted slope of the adiabatic atmosphere.

As a consequence of the decreased pressure at higher altitudes, cooking a meal requires different cooking times. Water boils at 100°C, but only at a pressure of

1 bar. In fact, the boiling temperature is a function of the pressure: with lower pressures, the boiling temperature drops.

boiling temperature

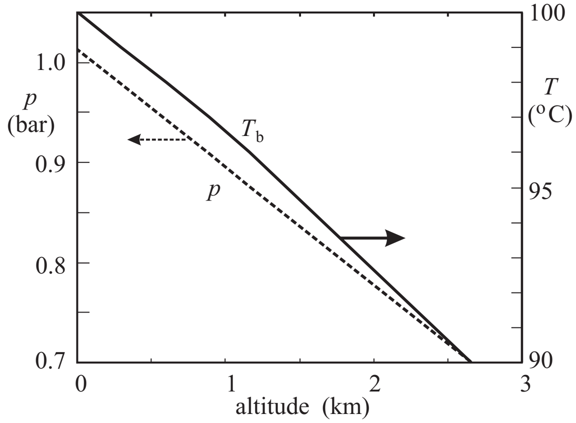

Figure 2.7 Pressure of the atmosphere is decreases with the altitude. Consequently, the boiling temperature of water drops with the altitude as well.

We can use a Taylor expansion of the pressure to find a linear approximation of the pressure. If we use the same numbers for pressure, density and temperature, we obtain:

$$p(z) \approx 101325(Pa) - 11.7756\left(\frac{Pa}{m}\right) * z \tag{2.20}$$

which shows that the pressure drops by approx. 11% per kilometer. So, when camping at an altitude of 2km, the pressure will be only about 0.8 bar. Consequently, the boiling temperature of water will have dropped to less than 93°C (see Figure 2.7).

pressure distribution

Moreover, at this altitude, the density of the air will have dropped from 1.2 kg/m^3 to 1.0 kg/m^3. Consequently, the amount of oxygen in a liter of air is 16% less than at sea-level. Thus, the gas-burner will not burn as hot and it is more difficult to keep the pan with the food at the desired temperature.
This lack of oxygen is also felt when hiking or doing any other form of physical exercise at high altitude. One is out of breath much quicker and it needs several days if not weeks before our body has adjusted to the new circumstances. At a height of 5km, i.e. in the Himalaya, the density has even decreased to 0.76 kg/m^3 and at the top of the world on Mt. Everest it is down to 0.5 kg/m^3: only 42% of what we are used to.

2.2 The earth's temperature

Usually, our temperatures range from −20°C in winter time to +40°C during the summer. These temperatures are vital to life as we know it. From a physics point of view, temperatures should be below 100°C as otherwise water is boiling and life as we know it is impossible. Moreover, they should not below zero be too long as then all water would freezes.

earth's temperature

We can make a rough estimate of why the earth's temperature is what it is. The earth is heated by the sun. It looses heat by radiating heat into the universe. The amount of heat received is in close balance with the amount released.

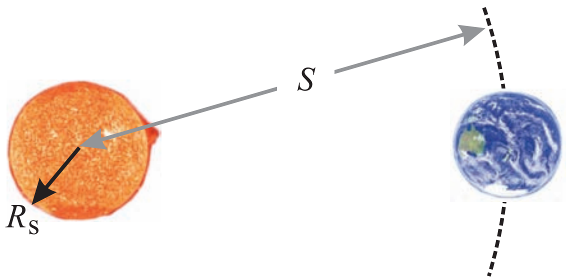

Figure 2.8 Sun heating the earth.

The most elementary model for predicting the earth's temperature just equates the amount of energy received by the earth from the sun to the amount the earth itself radiates. In a steady state, no heat is accumulated by the earth and the in-flow of energy from the sun will balance the out-flow. The former is given by considering the fraction of the total energy-flow from the sun that hits the earth. The sun radiates in all directions, the total flow is:

$$\phi_{sun} = 4\pi R_s^2 \cdot \sigma T_{sun}^4 \tag{2.21}$$

The earth circles the sun at a distance S, its projected area seen by the sun being πR_{earth}^2. Hence of the total radiation emitted by the sun, the earth receives:

$$\phi_{\text{from sun}}^{in} = \frac{\pi R_{earth}^2}{4\pi S^2} 4\pi R_s^2 \cdot \sigma T_{sun}^4 \tag{2.22}$$

black body radiation

Considering the earth in a first approximation as a black body, the total radiation emitted by the earth is:

$$\phi_{\text{earth}}^{out} = 4\pi R_{earth}^2 \cdot \sigma T_{earth}^4 \tag{2.23}$$

Now, using the energy balance for the earth in steady state, we get:

$$0 = \phi_{\text{from sun}}^{in} - \phi_{\text{earth}}^{out} \rightarrow T_{earth} = \left(\frac{R_{sun}}{2S}\right)^{1/2} T_{sun} \tag{2.24}$$

Wien's law

We can estimate the temperature of the sun from its spectrum: its maximum is around yellow light, *i.e.* $\lambda \approx 500nm$. From Wien's law we find for the temperature of the sun: T_{sun} = 5800 K. If we insert the relevant numbers in eq.(2.24): T_{sun} = 5800 K, $R_{sun} = 7.0 \cdot 10^8$ m and $S = 1.5 \cdot 10^{11}$ m, we find $T_{earth} = 280\text{K} = 7°C$, which is surprisingly close to the mean temperature of 15°C!

2.2.1 The greenhouse effect

The estimate of the earth's temperature fails on two points. Firstly, the earth is not a black body. It does not absorb all radiation from the sun, but reflects a rather significant portion. Secondly, there is the greenhouse effect.
The model for the earth's temperature can now be refined by taking into account the reflection of sunlight from the earth. The earth albedo a is about 0.3. Hence, the steady state balance (2.24) is modified by a factor $(1-a)$ in front of the incoming radiation. If we still assume that the earth radiates as a black body, the earth's temperature is reduced by $(1-a)^{1/4}$. This gives $T_{earth,albedo}$ = 253K. Now, the outcome is too low. *albedo*

Greenhouse effect
So far, we did not take into account the effect of the earth's atmosphere. That is to say, we ignored the greenhouse effect. This causes the earth to be significantly warmer. A simple model to incorporate this is sketched in Figure 2.9. *greenhouse effect*

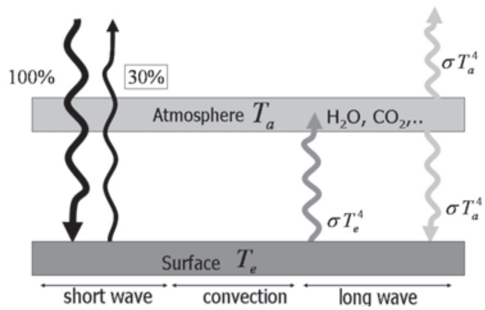

Figure 2.9 Schematic representation of the greenhouse effect.

Notice that roughly 30% of the incoming energy is immediately reflected (from the soil, seas and clouds). However, in a first approximation the atmosphere acts like the glass of a greenhouse. It captures the radiation from the earth en radiates as a black body with its own temperature, T_a. This radiation goes in two directions: into the universe but also back to the earth. From this sketch, a steady state balance for the earth & atmosphere can be set up:

$$0 = (1-a)\phi^{in}_{\text{from sun}} - 4\pi R^2_{earth}\sigma T_a^4 \tag{2.25}$$

On the other hand, for the earth itself also a steady state balance holds, but now with the radiation from the atmosphere as an incoming flow of energy:

$$0 = (1-a)\phi^{in}_{\text{from sun}} + 4\pi R^2_{earth}\sigma T_a^4 - 4\pi R^2_{earth}\sigma T^4_{earth} \tag{2.26}$$

Intermezzo: Black body radiation

All physical bodies loose heat via radiation. An ideal radiating body is the so-called black body. This is a body that absorbs all radiation that falls on it. Moreover, it radiates itself an amount of energy per unit time, that is proportional to its surface and to its temperature to the fourth power.

The radiation consists of electromagnetic waves, of which light is a particular example. A black body of temperature T emits an energy-flux according to Planck's law. This law describes the distribution of the wave length of the radiation, *i.e.* the spectrum, as a function of the temperature of the black body. In Figure 2.10, two examples of spectra are shown for a black body of 500°C and 800°C, respectively. From Planck's law a simple law has been derived, describing the wave length at which the maximum amount of energy is emitted (*i.e.* the 'top' of the spectrum). This is Wien's displacement law:

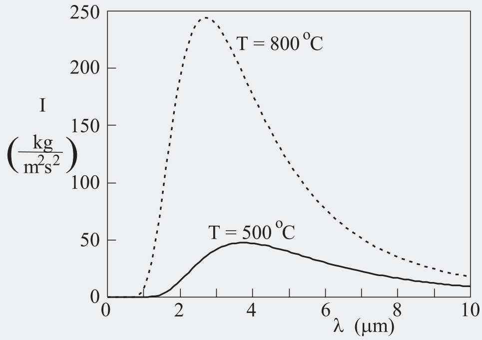

Figure 2.10 Spectrum of black body radiation.

$$\lambda_{max} T = 2.8978 \cdot 10^{-3}\,(mK) \tag{2.27}$$

with λ_{max} the wave length at which the spectrum has its maximum. Note that the horizontal axis of Figure 2.10 is in micro-meters, showing that the bodies radiate predominantly in the infra-red region, invisible for humans, but easy to pick up with infra-red detectors.

From Wien's law, we can easily compute the dominating wave length at which humans radiate. Our skin temperature is around 30-37°C = 303-310K. Hence, we radiate around 9.4 μm (compare to visible light with wave length of 0.5 μm). The sun has a surface temperature of about 6000K, hence its dominating wave length is some 0.5 μm which is yellow light.

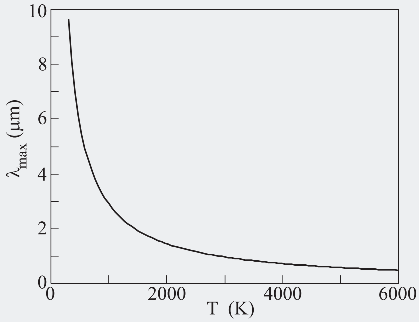

Figure 2.11 Wien's law.

The total energy emitted per unit area and per unit time, ϕ''_{rad}, of a black body is given by a simple equation, the Stefan-Boltzmann law:

$$\phi''_{rad} = \sigma T^4 \tag{2.28}$$

with $\sigma = 5.67 \cdot 10^{-8} W/m^2K^4$, called the Stefan-Boltzmann constant.

Intermezzo: Max Planck

Around 1900, Max Planck (1858-1947, Nobel Price for physics in 1918) made several important discoveries that paved the way for modern physics, in particular for Quantum Theory. He showed that light can not only be described as waves, but that it also appears as discrete amounts, quanta. Otherwise it would have been impossible to correctly describe observed phenomena like black body radiation. Before Planck's theory, it was believed that a correct account of the electrical, optical, and thermal properties of matter was possible on the basis of Newtonian's classical mechanic applied to motion and of Maxwell's theory of the electromagnetic field.

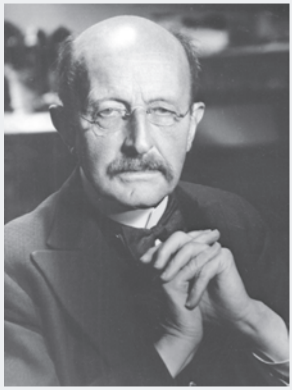

Figure 2.12 Max Planck (1858-1947)

However, Planck showed that this did not lead to a proper description of the law of heat radiation. Instead, it was necessary to introduce the quantum hypothesis, which has since received brilliant confirmation. This notion forms an important starting point for the development of Quantum Mechanics, that describes the 'world of the small' with spectacular accuracy, while at the same time turning the world of classical mechanics, with its deterministic view on the world, up side down.

Around the same time as Max Planck was Wilhelm Wien (1864-1924, Nobel Price for physics in 1911) working on heat and electromagnetism. Wien's theory of black body radiation was accurate for high frequencies, but showed discrepancies for the longer wave length. From his work he could deduce what is now known as Wien's law: the wavelength at which a black body radiates with maximum intensity is inversely proportional to the absolute temperature of the body. However, his theory broke down at short wave length. This formed an important starting point for Planck, which led Planck towards his quantum theory of radiation.

Figure 2.13 Wilhelm Wien (1864-1928)

On summing up the last two equations, the radiation from the atmosphere can be eliminated and the temperature of the earth can be resolved:

$$T_{earth} = [2(1-a)]^{1/4} \left(\frac{R_{sun}}{2S}\right)^{1/2} T_{sun} = 304K \tag{2.29}$$

So, the greenhouse effect significantly raises the temperature of the earth.

The temperature of the planets

temperature planets A similar exercise can be performed for all the other planets in our solar system. The calculations are done (i) for black bodies (T_{bb}), (ii) taking into account the planets albedo (T_{alb}) and (iii) assuming an atmosphere that provides a greenhouse effect (T_{gh}, even if the planet has no atmosphere). In Table 2.1 the relevant properties as well as the outcome of the calculations is given (together with the true temperature of each planet).

Planet	Distance from sun (10^6km)	albedo (-)	T_{bb} (K)	T_{alb} (K)	T_{gh} (K)	T_{true} (K)
Mercury	58	0.058	450	443	528	440
Venus	108	0.71	330	242	288	737
Earth	150	0.33	280	253	304	288
Mars	228	0.17	227	217	258	208
Jupiter	778	0.73	123	89	105	163
Saturn	1430	0.76	90	64	76	133
Uranus	2870	0.93	64	33	39	78
Neptune	4500	0.84	51	32	38	73
Pluto	5900	0.14	47	43	51	48

Table 2.1 Temperature of the planets

As can be seen from the table, the calculations give a reasonable result with Venus as the exception as it has an atmosphere with a strongly enhanced greenhouse effect. Further note, that the prediction for Mercury without greenhouse effect is almost right on the spot. This makes sense, as the sun has blown away Mercury's atmosphere.

Intermezzo: Grey bodies and albedo

The concept of the black body is an idealization of reality. Many objects do not obey the black body radiation law. In fact, they emit less energy than a black body and they absorb less radiation as well. To model reality better, grey bodies are introduced. They emit a similar spectrum as the black bodies, but with a reduced intensity. This reduction is called the emission coefficient, e. The total energy emitted per unit area and per unit time, $\phi''_{rad,grey}$, of a grey body is given by a simple equation:

$$\phi''_{rad} = e \cdot \sigma T^4 \qquad (2.30)$$

The reflection of the body is accounted for via the *albedo*, a. The amount of radiation reflected is a times the amount coming in; the amount absorbed is e times the latter. Hence, for a given spectrum (*i.e.* a given 'temperature' of the radiation), the albedo and emissivity add up to one: $a + e = 1$. However, a body radiates at its own temperature and may receive radiation from e.g. the sun. In that case a does not have to equal $1 - e$.

2.3 The blue sky

2.3.1 Atmospheric absorption

Light from the sun (and stars) will have to travel through the atmosphere before reaching the ground level. On its way it will be subject to absorption and scattering. How much of the sunlight is absorbed? Obviously, this depends on the time of the day, as at noon the sun is overhead and travels a much smaller distance through the air to us. The absorption is due to the air molecules and to 'dust' particles floating in the atmosphere. Figure 2.14 shows the difference between sunlight outside the atmosphere and at ground level as observed at sea level with the sun at 20° altitude.

The absorption is a function of the wave length of the light. Not surprising as atoms and molecules absorb light in specific frequency bands. *absorption of light*

2.3.2 Blue is the sky

Absorption is not the entire story. Sunlight (or light from the moon) also gets scattered by the air and the dust. This effect is so common, that hardly anybody pays attention to it. Nevertheless, it is an important feature of the atmosphere. When on a clear day you look into the sky its color is blue, everybody knows that. But few people know why. The reason is found in the scattering properties of the molecules: the probability of light being scattered by an air molecule is proportional to the wave length of the light to the power -4, or rephrased: proportional to f^4 (f the frequency of the light, the theory of molecular scattering was given first given by Lord Rayleigh). Thus, blue light of a wavelength of 450nm is compared to red light (λ = 650nm) $(650/450)^4 = 4.4$ times more likely to be scattered. Consequently, the blue end of the spectrum of the (white) sunlight has *light scattering*

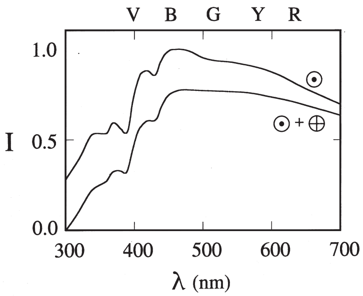

Figure 2.14 Intensity of sunlight outside the atmosphere (⊙) and at sea level (⊙+⊕, from [5]).

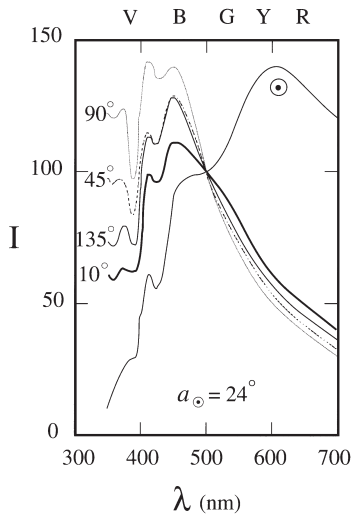

Figure 2.15 Comparison of spectrum of sunlight (⊙) and of sky light. The sun is at an angle of 24°. The spectra of the sky light are taken at various angles from the sun. The intensity of all spectra have been rescaled to the same value at $\lambda = 500nm$. Figure taken from [5].

a reduced probability to reach our eye directly in comparison with the red end. And thus most of the scattered light that reaches us is blue: the sky is blue. This is illustrated in Figure 2.15), that compares the spectrum of light coming directly from the sun to light coming from the sky (called sky light).

2.3.3 Sky light

brightness of the sky

The sky does not have a uniform brightness. This is easily verified when looking on a bright day into the sky. The sky above us is 'deep' blue. Towards the horizon the sky becomes lighter. Figure 2.16 shows an example.
The effect is caused by the difference in the length of the line of sight when looking at different angles into the sky. Directly overhead (this is called the Zenith) the distance through the atmosphere is smaller, see Figure 2.17.

Figure 2.16 Blue sky: with dark blue overhead and lighter close to the horizon (Columbine Lake, Colorado).

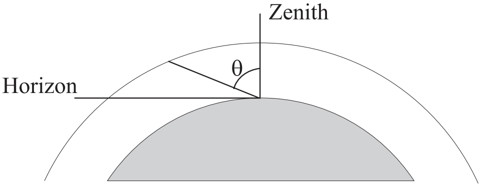

Figure 2.17 Geometry of the line of sight through the atmosphere when looking towards space.

Intermezzo: Lord Rayleigh

Lord Rayleigh (1842-1919), born as John William Strutt, is well known for his contribution to our understanding of optics and vibrations, including sound. His work covers almost the entire field of physics, including sound, wave theory, color vision, electrodynamics, electromagnetism, light scattering, flow of liquids, hydrodynamics, density of gases, viscosity, capillarity, elasticity, and photography. He received the Nobel price in 1904.

Figure 2.18 Lord Rayleigh (1842-1919)

When looking at an angle θ with the vertical we see through more air. Thus, more light will be scattered into our direction and the sky looks brighter. As a first approximation the ratio of the amount of air viewed in direction θ compared to $\theta = 0$ increases as $1/\cos(\theta)$. This rule brakes down at large angles, when the curvature of the earth becomes important. The ratio is plotted in Figure 2.19, for a flat earth (when the line of sight can become infinitely long) and the actual spherical geometry of the earth.

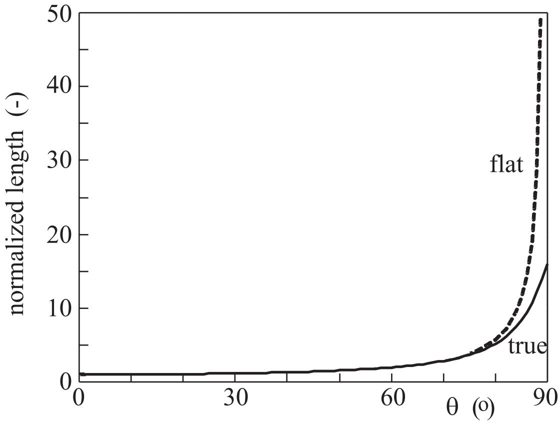

Figure 2.19 Length through the atmosphere as a function of the zenith angle θ, normalized by the length at $\theta = 0$

.

Note that in Figure 2.16 close to the horizon the sky is becoming very light blue. The reason is that we - wrongly - assume that the more air is on the line of sight the more (blue) light gets scattered to us. We have ignored multiple scattering. And this causes a very thick atmosphere to be opaque.

A simple model for the brightness of the sky itself can be conceived as follows. The line of sight is in the direction x, making an angle θ with the vertical. For simplicity, assume that the sun is straight above us (at $\theta = 0$), then what we see in the direction x is scattered sunlight, i.e. no direct sunlight. In Figure 2.20 the situation we want to analyse is shown.

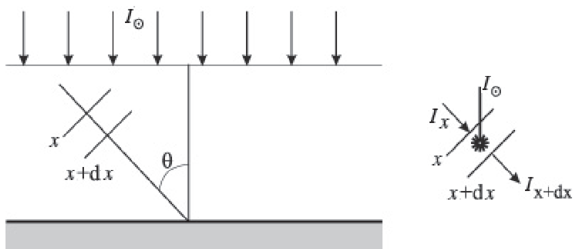

Figure 2.20 Schematic representation of sky light received by the observer.

Intermezzo: Scattering of light by molecules

Light traveling through the atmosphere will have interaction with the molecules of the air. One of the possible interactions is scattering. This can be understood by considering a simple molecule made of a fixed nucleus with one electron orbiting it. The equation of motion of the electron can be written as that of a harmonic oscillator, with eigen frequency ω_0:

$$m\ddot{x} + kx = 0 \rightarrow \ddot{x} + \omega_0^2 x = 0 \tag{2.31}$$

When light passes the electron, a force acts upon the electron (since light is an electro-magnetic wave). The electric field is the dominating force. For light of wave length λ, *i.e.* angular frequency $\omega = 2\pi f = 2\pi\frac{c}{\lambda}$, the electric field can be written as $E_0 \sin\omega t$. Such a field will produce a force $F_e = eE_0 \sin\omega t$ on the electron, modifying its equation of motion to:

$$\ddot{x} + \omega_0^2 x = \frac{e}{m} E_0 \sin\omega t \tag{2.32}$$

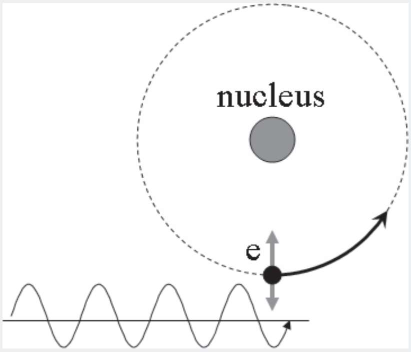

Figure 2.21 Simple model of light having interaction with an atom.

The solution to this equation is of the form:

$$x(t) = c_1 \sin\omega_0 t + c_2 \cos\omega_0 t + \frac{eE_0}{m}\frac{\sin\omega t}{\omega_0^2 - \omega^2} \tag{2.33}$$

The important part is the last term: the extra motion caused by the passing electric field. This causes an additional acceleration of the electron: $a(t) = -\frac{eE_0}{m}\frac{\omega^2}{\omega_0^2-\omega^2}\sin\omega t$. The electron in its original orbit does not radiate. But, due to the extra acceleration the electron starts radiating. It sends out an electromagnetic field with the wave length of the incoming light and an intensity proportional to the square of the acceleration, $\langle a(t)^2 \rangle$, *i.e.*

$$I \propto \left[\frac{\omega^2}{\omega_0^2 - \omega^2}\right]^2 \tag{2.34}$$

As the eigen frequency ω_0 of the electrons in oxygen and nitrogen is much higher than the frequency ω of the incoming light we have that this is basically proportional to $\left(\frac{\omega}{\omega_0}\right)^4$. As this radiation by the electron obviously feeds on the incoming light, we find that the scattering of the light is proportional to the frequency of the incoming light to the power 4.

intensity on line of sight

Consider a part of the line of sight between $\{x, x+dx\}$. The intensity of the light is called I, the initial intensity outside the atmosphere is $I_{\odot}$. The light that is coming along the x-axis into the observer's direction has an intensity I_x at position x, when it is at position $x+dx$ its intensity has become I_{x+dx}. This change is caused by scattering. In a good approximation the loss is proportional to the intensity of the incoming light, so μI_x is scattered in different directions and will no longer reach the observer. But, we shouldn't forget the light that is added to our line of sight from direct sunlight that is scattered into our direction. Again this is proportional to the intensity of the sunlight at position x. We can safely ignore that this is, due to scattering, no longer the original intensity $I_{\odot}$ and we formally write for this gain-term μI_0. Note, that we did not use in this last expression $I_{\odot}$. This is because we now deal with scattering of light by the molecules on the line segment dx precisely into our direction, whereas to account for the losses all we need is that the light originally moving in our direction is being scattered away from the line of sight, without having to specify into which direction. So, I_0, is taking all this into account and is a lumped (unspecified, but constant) parameter.

Now obviously, we should also take into account that the loss and gain term are also proportional to the number of molecules on the small piece dx. As discussed before, the density of the atmosphere is a function of the altitude. In order to keep things simple, we will ignore this and consider the density as a constant. Consequently, the number of molecules is proportional to the length of the line segment, dx, we are considering. The longer dx, the more air molecules will participate in the scattering. If we add everything up, we find:

$$0 = I_x - I_{x+dx} - \mu I dx + \mu I_0 dx \tag{2.35}$$

If we use a Taylor expansion on I_{x+dx}:

$$I_{x+dx} = I_x + \frac{dI}{dx}dx \tag{2.36}$$

we can reduce eq.(2.35) to

$$\frac{dI}{dx} = \mu\left(I_0 - I\right) \tag{2.37}$$

The general solution is:

$$ln\left(I_0 - I\right) = -\mu x + C \tag{2.38}$$

The integration constant C is found from the boundary condition. At position $x = 0$, which is the outer boundary of the atmosphere, no sunlight is yet reflected into our line of sight as no molecules are present there. Thus, we find $C = lnI_0$ and the solution for the brightness of the atmosphere in direction x is:

$$\frac{I(x)}{I_0} = 1 - e^{-\mu x} \tag{2.39}$$

Note that μ has a dimension of m^{-1}, *i.e.* it is the reciprocal of a characteristic distance Λ. The observer will see, of course, the intensity at position x_{max} which denotes ground level. The length through the atmosphere changes with the angle, θ with the vertical as $H/\cos(\theta)$ (H is the thickness of the atmosphere above us). This ignores the curvature of the earth for which we can adjust. So, the sky light intensity that we see when looking into different directions is given by (the earth-atmosphere is approximated as a flat, infinitely wide system)

$$\frac{I(\theta)}{I_0} = 1 - e^{-\frac{H}{\Lambda \cdot cos\theta}} \tag{2.40}$$

For $\theta = 0$ we have $I(0)/I_0 = 0.1$ (see [5]). Thus $\frac{H}{\Lambda} = -ln(0.9)$. The brightness of the sky is plotted in Figure 2.22. The dashed line is the flat atmosphere approximation, the full curve the one adjusted for the earth curvature.

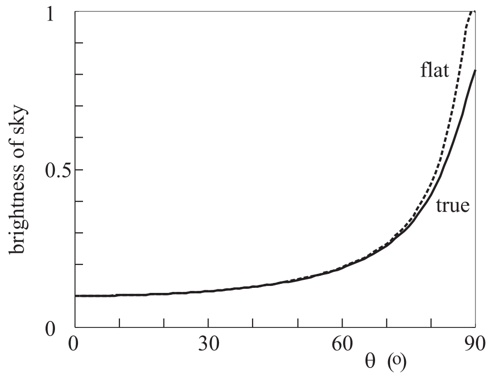

Figure 2.22 Brightness of the sky for an observer at sea level, viewing at angle θ with respect to the vertical.

Now we understand the change in brightness of the sky, but we haven't answered the question why the sky close to the horizon looses its blue color. The answer is simple. The sky light intensity saturates at large angles; it will do so for all wave lengths, so blue isn't the dominating color any more, but all other colors mix in and they do so at comparable intensities. Hence, the sky turns white.

2.3.4 Distant mountain appear lighter

Another consequence of the scattered sunlight that is related to the whiter color of the horizon, is seen when looking at a distant mountain ridge. The mountains furthest away seem lighter, those closer by look darker, see Figure 2.23.
The explanation is that the further away, the more scattered sunlight mixes in with the light that is directly coming from the mountains. Since a mountain close by hides some of the scenery behind it, the distance to that mountain top is significantly less than that of the scenery seen just above the ridge. In other words there is a discontinuity in the distance when looking from the mountain to the scenery behind it. This is illustrated in Figure 2.25. Consequently, the intensity from the scattered light changes rather abruptly.

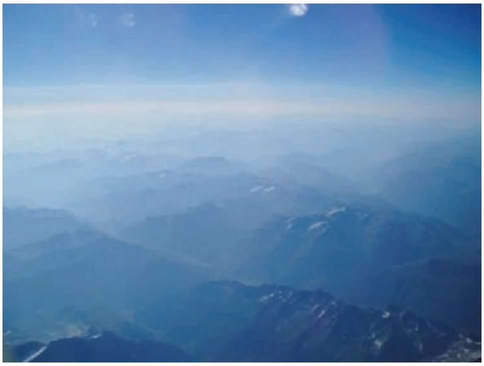

Figure 2.23 Distant mountains appear lighter than those close by.

Figure 2.24 Distant mountains (190 km away) not visible during the day, but showing clearly at sunset (from [5]).

The sky light can be so bright that mountains at great distances can not be seen during the day. But at sunrise or sunset they become visible as dark areas against a red sky, see Figure 2.24.

2.4 Color of smoke, fog or clouds

The color of cigarette smoke is different when being inhaled first compared to coming directly from the cigarette. Why? Again this can be explained by the scattering properties of the smoke. When rising up directly from the cigarette, the smoke particles are very small. Many of them are in fact smaller than the wave

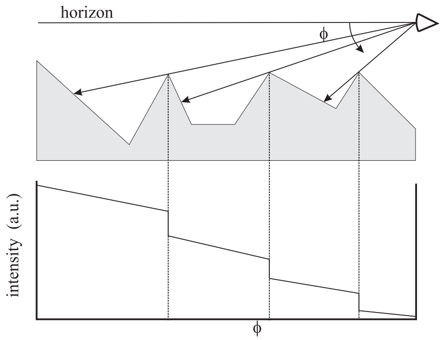

Figure 2.25 Discontinuity in brightness caused by discontinuity in distance seen.

length of visible light. This means that they have scattering properties quite similar to the air molecules discussed before. Therefore, blue light is preferentially scattered and the smoke looks blueish to us. If on the other hand, the smoke is first inhaled it contains relatively large water droplets. These are larger than the wave length of visible light and will therefore have different scattering properties (that can be described by the Mie theory, by Gustav Mie (1868-1957), explaining the scattering of light by spherical particles of any size). It means that the preference for scattering of higher frequencies is lost. The scattering is for the 'large' droplets much closer to ordinary reflection and refraction. This makes the smoke look white. Of course, this argument will also hold for fog or clouds.

Big or small?
One might wonder what makes the scattering different for small or big water droplets. After all, all the scattering is done by the water molecules which are the same in any type of droplet. Or perhaps, equally puzzling now that we start to think about it: why can clouds be seen anyhow? Before the cloud is formed the water vapor is present already at the location of the cloud-to-form. All that is needed is a sufficient decrease in temperature (and some nuclei to start the condensation process). The total number of water molecules on the line of sight hasn't changed at all! The difference is in the scattering of single isolated molecules (the vapor) and an agglomerate of atoms (a droplet). In the droplets the atoms are packed together at inter-molecule distances much smaller than the wave length of the light. Now, remember that light is 'nothing' but electro-magnetic waves. The molecules respond to the passing electro-magnetic field. In case of the isolated molecules they do so individually and the scattering is proportional to N, the number of molecules. But when two molecules are very close together, they start to respond in phase to the electro-magnetic field. This gives a summation of the contributions of each of the two molecules to the amplitude of the scattered electro-magnetic field, which means that the energy of the scattered light is *four* times higher, as the energy is proportional to the square of the wave amplitude. Thus, the linearity in N is broken. The conclusion is that scattering is stronger by

light scattering at droplets

lumps of water molecules than by separate molecules. Thus, water vapor is invisible to the naked eye, but droplets are easily seen. For a more complete description of the scattering see 'The Feynman Lectures on Physics', part I Ch.32 ([1]).

2.4.1 Clouds: floating droplets

Archimedes' law

Clouds consist of 'floating' water droplets. What is suspending them in the air? Obviously, gravity is pulling them down. And the upwards directed buoyancy force of the air (Archimedes' law: the upward force on an object submerged in a fluid is equal to the weight of the displaced fluid) is insufficient to keep the droplets up. So, apparently there is some force acting on the droplets in the upward vertical direction.

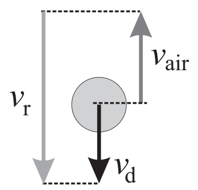

Figure 2.26 Relative velocity between droplet and flowing air.

This force comes from the interaction between the droplet and the surrounding air. If there is a relative motion between these two, the droplet feels a frictional force acting in the opposite direction of the relative motion. This force, usually called the *drag force*, is proportional to the magnitude of the relative velocity squared, the density of the surrounding air and the area of the projection of the droplet perpendicular to the velocity, $A_\perp$. The relative velocity, $\vec{v}_r$, is the difference between the velocity of the droplet, $\vec{v}_d$, and that of the surrounding air, $\vec{v}_{air}$, i.e.:

$$\vec{v}_r = \vec{v}_d - \vec{v}_{air} \tag{2.41}$$

drag force

Furthermore, this drag force (see intermezzo) is in the opposite direction of the relative velocity. So we have:

$$\vec{F}_D = -C_D A_\perp \frac{1}{2} \rho_{air} |\vec{v}_r|^2 \frac{\vec{v}_r}{|\vec{v}_r|} \tag{2.42}$$

Let's assume that the vertical component of the velocity of the water droplets in a fog or cloud is zero. We then have that the vertical component of the relative velocity is equal to $-v_{air,z}$. If we assume further that the horizontal velocity of the air and the droplets is the same, we have a non-zero vertical component of the drag force. It si given by:

$$F_{D,z} = -C_D A_\perp \frac{1}{2} \rho_{air} |\vec{v}_{air}|^2 \frac{-v_{air,z}}{v_{air,z}} \tag{2.43}$$

Obviously, gravity is pulling the droplets downwards, so the drag force must have a vertical component that is pointing upwards. This can only be so if the vertical component of the air velocity is pointing upwards! Apparently, the motion of air in a cloud has an upward component that is sufficient to keep the water droplets 'floating': the drag force exerted by the air flowing around the droplet balances the pull of gravity. In Figure 2.27 the air velocity needed is given as a function of the size of the water droplets.

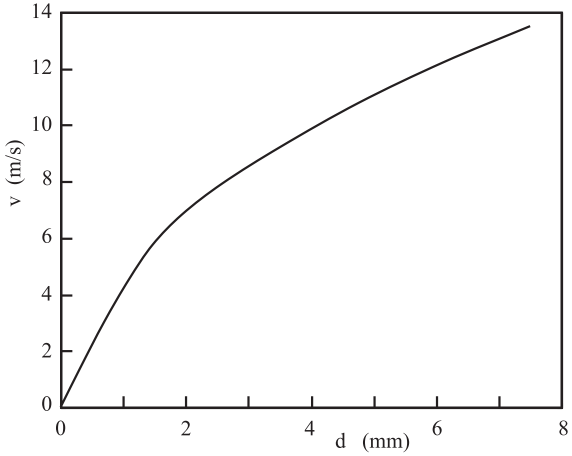

Figure 2.27 Required air velocity for suspending a water droplet as a function of the droplet diameter.

The atmosphere is always in motion. It is by and large transparent for sunlight. So, if there are no clouds most of the sunlight reaches the ground and heats it up. The ground in turn heats up the air. This process obviously induces temperature and density differences, causing e.g. upward motion of hot air. In a cloud the droplets may grow until finally their size is such, that the force of gravity, which is proportional to the diameter to the third power, is larger than the drag force that goes as the square of the diameter. Then the droplets will fall and we have rain, or hail etc. Even during their fall to the earth the droplets may grow. Furthermore, the upward velocity of the air close to the ground could be small or zero. In the latter case we find that the velocity of a droplet with a size of 3mm is approximately 8m/s.

- *Example: What is the speed of a returning bullet when shooting in the air?*

This question was raised in news on tv at the time freedom fighters in Albania were celebrating their victory. The estimates that followed on television and in the newspapers were all dangerously high: several hundreds of meters per second up to a kilometer per second. This would certainly be deadly if hit by such a returning bullet.

Intermezzo: the drag force

Any object moving at a relative speed in a continuous medium (either a gas or a fluid) will experience a drag force that tries to make it move at the velocity of the medium. The drag force consists of two contributions. One is the viscous drag due to the difference between the velocity of the surface of the object and the medium. The other is due to a pressure difference that builds up over the object: an increased pressure at the upstream side of the object as the fluid has to be 'stopped' and pushed aside (this requires a force which shows up as an increased pressure) and a decreased pressure at the back of the object, the region called its wake, e.g. due to an 'overshooting' of the fluid.

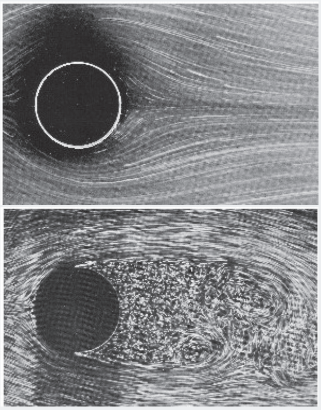

Figure 2.28 Flow lines around a cylinder at low velocity (top) and high velocity (bottom).

The general modeling of the drag force is based on the notion that the bigger the object obstruction is, the higher the drag force. Note that both the actual surface (for the viscous drag) as well as the projected area on which effectively the pressure difference works play a role. The choice has been made to use the projected area as in most cases the pressure difference is the deciding contribution. A first estimate of the pressure difference is based on the so-called pressure head $\frac{1}{2}\rho_{fl}v^2$, which is the pressure needed to stop an incoming flow of velocity v. So, the drag force is a pressure difference times an area. Everything that makes this first estimate incorrect is put in the 'proportionality constant', C_D, called the drag coefficient:

$$F_D = -C_D A_{\perp} \frac{1}{2} \rho_{fl} v^2 \tag{2.44}$$

The constant C_D is not really a constant, but rather a function of the Reynolds number, $Re \equiv \frac{\rho v D}{\mu}$, the single most important dimensionless number in fluid mechanics (with ρ the density of the fluid, v the relative velocity, D the characteristic dimension of the object and μ the dynamic viscosity of the fluid). Low Reynolds numbers indicate laminar flows, high Reynolds numbers turbulence.
Only for simple objects (like spheres) and low Reynolds numbers (around $O(1)$) is the drag coefficient analytically known. For all other cases we rely on experimental data. The figure below gives the drag coefficient for a few simple objects as a function of the Reynolds number.

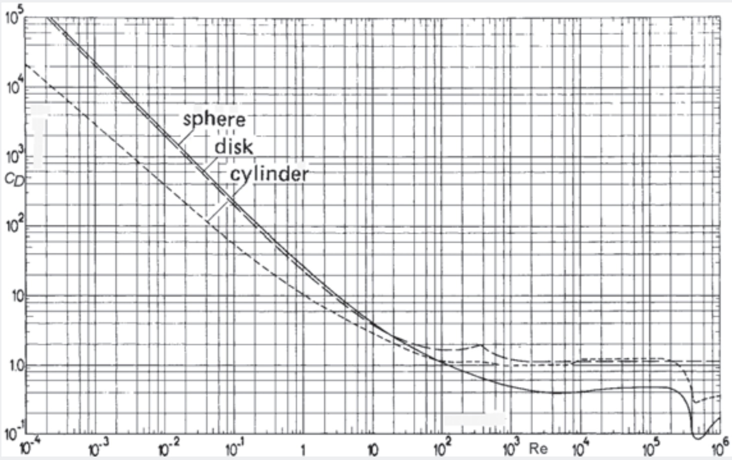

Figure 2.29 Drag coefficient as a function of the Reynolds number.

A falling droplet

Consider a spherical rain drop of 1mm diameter. The drop is falling at constant velocity through quiet air. What is the velocity of this droplet?

To answer this question, we set up, according to Newton's laws, a force balance. Three forces are acting on the drop: gravity downwards, the buoyancy force and drag force upwards:

$$0 = F_D + F_b - F_g = C_D \frac{\pi}{4} D^2 \frac{1}{2} \rho_{air} v^2 + \rho_{air} \frac{\pi}{6} D^3 g - \rho_{dr} \frac{\pi}{6} D^3 g \tag{2.45}$$

where $D = 1$ mm the droplet diameter, ρ_{air}= 1.2 kg/m^3 and $\rho_{dr} = 1.0 \cdot 10^3$ kg/m^3 the density of the air and the droplet, resp. From the above equation we can not solve v, as we need to know C_D, which is an unknown function of the Reynolds number that depends on v.

The way to solve this, is using an iterative method in which we first guess C_D, then calculate v. Next from v, we calculate Re (with $\mu = 2.0 \cdot 10^{-5}$kg/ms the viscosity of air) and look up the new value for C_D in the figure above and repeat the sequence until convergence is reached. For the example this means (according to eq.(2.45)):

$$v = \sqrt{\frac{4}{3} \frac{gD}{C_D} \frac{\rho_{dr} - \rho_{air}}{\rho_{air}}} \tag{2.46}$$

$$\begin{array}{llll}
\text{guess:} & C_D = 0.43 & \rightarrow v = 5.0m/s \rightarrow Re = 3.0 \cdot 10^2 \\
\rightarrow & C_D = 0.7 & \rightarrow v = 3.9m/s \rightarrow Re = 2.4 \cdot 10^2 \\
\rightarrow & C_D = 0.8 & \rightarrow v = 3.7m/s \rightarrow Re = 2.2 \cdot 10^2 \\
\rightarrow & C_D = 0.8 & \rightarrow \text{conv.: the answer is: } v = 3.7m/s
\end{array}$$

Figure 2.30 Shooting bullits in the air.

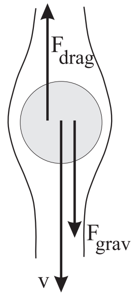

Figure 2.31 Gravity and drag acting on a falling sphere.

The real velocity is much lower. Let's assume that the bullets are small steel spheres with a diameter of 5 mm. Gravity will act on them. If this was the only force, things would be easy. Assuming the bullet to be fired perfectly vertically upwards the only parameter determining the return velocity is the velocity with which the bullet leaves the rifle. Just for the sake of the argument let's assume that this is 600m/s. As only gravity acts, the bullet's total energy (i.e. kinetic and potential energy) is conserved. Hence, if the bullet returns, its potential energy is the same as when it left and consequently its kinetic energy is at its starting value. Thus the velocity when returning would indeed be 600m/s.

But of course gravity is not the only force. On earth an object usually travels through air or water. Thus, we should include the frictional force exerted by the air on the bullet. In a steady state the gravitational force is balanced by the drag, ignoring buoyancy (see Figure 2.31.

Thus we have that the net force on the bullet is zero and therefore its velocity is constant:

$$0 = F_{drag} - F_{grav} = C_D A_\perp \frac{1}{2} \rho_{air} v^2 - mg \tag{2.47}$$

Like in the previous example, with $A_\perp = \frac{\pi}{4} d_b^2$ and $m = \frac{\pi}{6} \rho_b d_b^3$, we find for the velocity of the bullet:

$$v_b = \sqrt{\frac{4}{3} \frac{\rho_b}{\rho_{air}} \frac{g d_b}{C_D}} \tag{2.48}$$

where d_b denotes the bullet diameter and ρ_b the density of the bullet. Substituting numbers: $d_b = 5$ mm, $\rho_b = 8.9 \cdot 10^3$ kg/m^3, $\rho_{air} = 1.2$ kg/m^3 and taking a value of order 0.5 for the drag coefficient, we find for the velocity : 31 m/s.

bullet free fall velocity

Still a hit would hurt, but it is probably not fatal! Notice that the initial velocity at which the bullet was fired is irrelevant (provided of course it was well above the velocity we just calculated).

2.4.2 Staring in the fog

Clouds and fog are very similar: a large number of floating small water droplets. The droplet size is sub-millimeter as otherwise the terminal velocity of the droplets is so high that a fog can quickly disappear (see Figure 2.27). Some fog is ’open’ and one can look several hundred meters ahead. Others are dense and the view is limited to less than a few meters. What causes this difference? There are a few properties that are important: the size of the droplets, d_{dr}, and the number of droplets per unit volume, n_{dr}. Obviously, the higher the number of droplets per unit volume the more difficult it is to see through. In Figure 2.32 a sketch is made of light traveling through a foggy area.

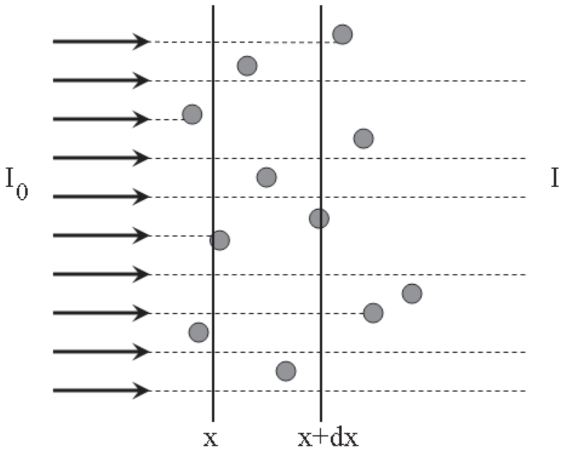

Figure 2.32 Light penetrating into the fog.

light penetration in the fog

The incoming light has an intensity I_0. It passes through a fog, formed by spherical droplets. If we consider the difference in area-averaged intensity between the light passing at a plain at position x and $x+dx$, the latter will be reduced due to the blocking of the droplets of part of the light rays. We can make a simple model of this effect. The reduction in intensity in the slab $\{x, x+dx\}$ is proportional to the incoming intensity, $I(x)$. Moreover, it is proportional to the projected area of the droplets in this slab, $\sum \frac{\pi}{4} d_{dr}^{d}$:

$$I(x+dx) = I(x) - I(x) \cdot \frac{\sum \frac{\pi}{4} d_{dr}^2}{A_{slab}} \tag{2.49}$$

The total blocking area in the slab can also be written as $N \cdot \frac{\pi}{4} d_{dr}^{d}$, with N the total number of droplets in the slab. We will replace $\frac{N}{A_{slab}}$ by $n \cdot dx$, with n the number of droplets per unit volume. A related measure of the number of droplets per unit volume is the volume fraction of droplets, α, defined as:

$$\alpha \equiv \frac{V_{dr}^{tot}}{V^{tot}} = \frac{N \cdot \frac{\pi}{6} d_{dr}^3}{V^{tot}} = n \frac{\pi}{6} d_{dr}^3 \tag{2.50}$$

Combining the equations (2.49) and (2.50) we find:

$$I(x+dx) = I(x) - I(x) \cdot \frac{\alpha \frac{\pi}{4} d_{dr}^2}{\frac{\pi}{6} d_{dr}^3} dx \tag{2.51}$$

which can be rewritten as:

$$\frac{I(x+dx)-I(x)}{dx} = -\frac{3}{2}\frac{\alpha}{d_{dr}}I(x) \rightarrow \frac{dI}{dx} = -\frac{3}{2}\frac{\alpha}{d_{dr}}I \tag{2.52}$$

Solving eq.(2.52), with boundary condition $x = 0 \rightarrow I = I_0$ gives:

$$I(x) = I_0 \exp\left(-\frac{3}{2}\frac{\alpha \cdot x}{d_{dr}}\right) \tag{2.53}$$

The reason to use the droplet volume fraction rather than the number of droplets per unit volume is, that use of the volume fraction allows us to compare fog with tiny droplets and rain with relatively big ones at the same water content. The view in a typical rain shower is less obstructed than in a fog. This is a consequence of the size of the droplets. In a rain shower the droplets will have a diameter of about 2 mm (with terminal velocity of about 7 m/s).
Let's assume that the droplet number density is 100 droplets/m^3.

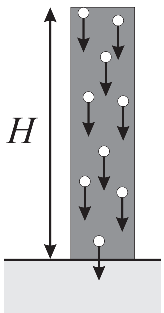

Figure 2.33 A column of rain droplets.

Figure 2.34 A weak sun seen through the fog.

Suppose that it would rain an hour and the water is not drained. The thickness of the water layer on the ground would then be 1 cm. This is easily calculated from the droplet size, d_{dr}, the number of droplets per unit volume, n_{dr}, and the terminal velocity of the droplets, v_{dr}. We could imagine that all rain drops that will fall during that hour are contained in a vertical column of air. The height of this column is $H = v_{dr}t$, with t the time interval it rains. In the present case this would be H = 25 km. The amount of water (in the form of the droplets) in this column is calculated from $V_w = \left(n_{dr}\frac{\pi}{6}d_{dr}^3\right)A \cdot H$, with A the cross-sectional area of the column. If all this water will form a layer, the layer height, h, follows from: $A \cdot h = V_w \rightarrow h = \left(n_{dr}\frac{\pi}{6}d_{dr}^3\right) \cdot H$ = 1cm in the present case. Note that $n_{dr}\frac{\pi}{6}d_{dr}^3$ is the amount of water in an unit volume, hence it equals the volume fraction α, which is in this example equal to $4.2 \cdot 10^{-8}$. So, we could write $h = \alpha H$.

Let's go back to the visibility question. From eq.(2.53) we will define the visibility length l as the distance at which the intensity has dropped to e^{-2} of the original one, *i.e.*

$$\frac{I(l)}{I_0} = e^{-2} \rightarrow l = \frac{4}{3}\frac{d_{dr}}{\alpha} \tag{2.54}$$

For the rain shower this means l = 6.4 km. Hence, the view is not significantly obstructed. If, however, the same amount of water was not in the form of droplets of 2mm, but of 0.2mm the answer would be l = 640m and the rain would be noticeable. A droplet size of 0.2mm is still too big for a good fog, as the steady state velocity of these droplets is about 0.6m/s.
A real fog would have even smaller droplets, let's say 20μm. Then, the terminal velocity is 1cm/s. Now we find for the visibility length: l=64m. Hence, this fog is getting dense.

So, small droplets have a big effect on the visibility. This is a consequence of keeping α, *i.e.* the amount of water per unit volume, constant. Then, by taking smaller and smaller droplets, we create an increasing blocking area as the surface/volume ratio goes up for smaller droplets. Actually, this ratio increases with d_{dr}^{-1} which is exactly the dependence we find in the visibility length.

Relation (2.53) is obviously an approximation. We assumed that light rays hitting a rain droplet are stopped and do no longer contribute to the light intensity further into the fog. Light gets scattered in other directions, the absorption is not big. The scattered light will get rescattered and penetrates into the fog. Moreover, we did not take into account that droplets deeper in the fog will be in the shadow of those closer to the light source. Thus our estimate of the blockage of the droplets by their frontal area is too optimistic. However, the multiple scattering and the overestimated blocking partially cancel each other out. This makes that our relation is not so bad after all as has been confirmed by experiments.

2.5 Color of the sun

When, on a clear day, we would look into the sun at the middle of the day, we would find that the sun is white. It is almost as white as it is seen in space. Even though some of the visible sun light is scattered by the atmosphere and some is absorbed, most of it is reaching us. The spectrum is not affected too much and thus the sun's color is white. However, when the sun is low, the sunlight has to pass through a much thicker layer of the atmosphere. This means scattering, which is especially effective on the high frequency part of the spectrum (remember the frequency to the fourth power in the scattering). Thus, the sunlight we receive from a low sun has lost quite some energy over all wave lengths and the sun is less bright.
But in addition the blue end of the visible spectrum is weakened more: the sun starts to look yellow. On top of the scattering is absorption by water vapor and

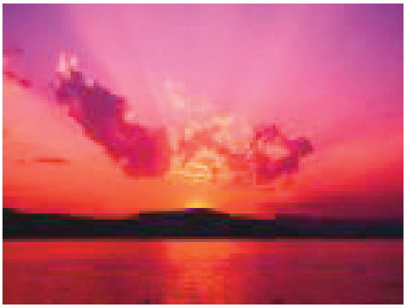

Figure 2.35 The evening sun coloring the sky red and purple.

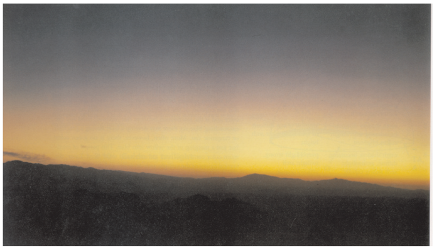

Figure 2.36 Twilight arch, just before sunrise (from [5]).

red setting sun

ozone, that absorbs extra in blue and green. The lower the sun, the longer the path through the atmosphere. So, finally the scattering of the yellow is also so effective, that the sun turns red . Here also small 'floating' particles, dust, smoke or small droplets (all called aerosols) help in scattering the light. If their size is in the order of 100 nm, they are especially effective. As their concentration (i.e. number per unit volume) differs from day to day, so is every setting sun different in color.

Twilight arch

yellow twilight arch

Just after sunset, the horizon colors yellow over a wide angle (see Figure 2.36). This is called the twilight arch. As the sun is just below the horizon, no direct sunlight can reach the observer. The sky above the horizon, however, is still illuminated by the sun. The observer will see this light indirectly, i.e. via scattering. The light path that the sunlight has to travel through the atmosphere before illuminating the sky above the observers horizon is quite long. So, most of the blue light will be scattered before it reaches that part of the atmosphere (see Figure 2.37).
This means that the scattered light is mostly yellow and that the twilight arch is yellow. As the sun sets more, the path through the upper sky above the horizon gets longer and thus the twilight arch turns redder and redder. At the same time the portion of the visible sky that is illuminated by the sun gets smaller and smaller: it gets dark. Note that in Figure 2.36 the twilight arch is brightest just above the horizon and gets redder when moving upwards. This is also a consequence of the length of the light path to the illuminate sky as well as the density of air molecules that scatter the light.
Due to symmetry, the twilight arch can of course also be seen just before sunrise. The part of the horizon opposite to the sun will be dark just before sunrise or just after sunset. This is caused by the shadow of the earth itself, that blocks the sun from illuminating the sky just above that part of the horizon. So, we will see a dark band just above the horizon. This is known as the *anti-twilight arch*.

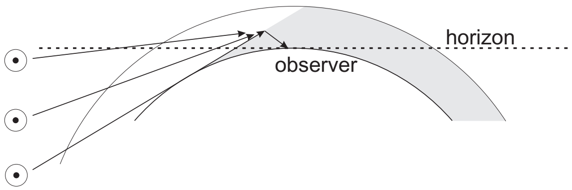

Figure 2.37 Light path length to the illuminated sky above the horizon increases as the sun sets more below the horizon. Furthermore, a smaller portion of the upper atmosphere can be seen by the observer.

Purple sky

When the sun is about 5° below the horizon, the sky some 45° above the sun's position may show a purplish glow, see Figure 2.38. This is called the 'purple light'. It most likely is caused by the scattering of light in the stratosphere (about 20-25 km high) by dust particles. Most of this light that reaches the observer is blue. However, it gets mixed by light that is coming from the lower atmosphere. This light is red, as discussed above. So, the observer sees a mixture of red and blue, i.e. purple. The different light paths are shown in Figure 2.40. The particle number density of dust (i.e. the number of particles per unit volume) in the stratosphere varies from time to time. Volcanic eruptions add dust to the stratosphere, and so do meteorites from space. This causes the occurrence and intensity of the purple light to change.

purple light

Figure 2.38 Purple light above the twilight arch (from [5]).

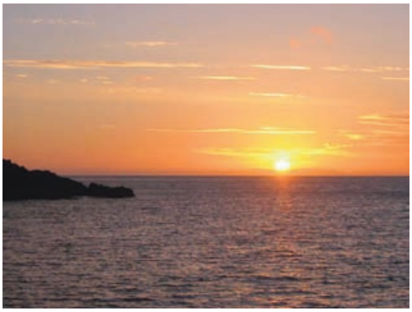

Figure 2.39 The setting sun, seen just above the horizon.

2.6 Where is the sun at sun setting?

If the sun gets very close to the horizon, its shape seems to change from a sphere to an ellipsoid which has a height-to-width ratio of less than one. This is caused by refraction of sunlight in the atmosphere. The refraction index, n, of the atmosphere is not a constant, but depends on the density and on the type of molecules.

refraction index

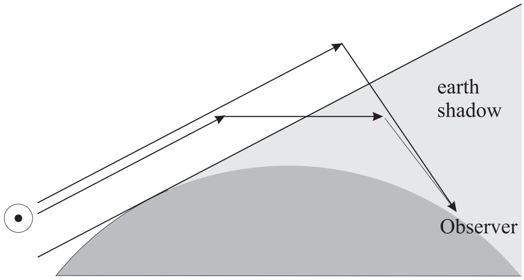

Figure 2.40 Different paths of sunlight to the observer lead to a mixture of red light that follows a path through the lower atmosphere and blue light scattered from the stratosphere.

In vacuum (e.g. outer space) its value is 1. In the atmosphere it gradually increases from a value of one at the 'edge' of the atmosphere towards a value of about 1.0003 at ground level (it depends on the wavelength and is slightly higher for the blue side of the visible spectrum than for the red side). The change might look small but the effects this has are not negligible. For instance, when the sun is seen to touch the horizon it actually is below the horizon! This can be easily understood if we think about the path of light beams through a medium that is increasing its refraction index along the beam path.

Snell's law

When a light beam passes an interface separating two layers with different refraction indices, we have according to Snell's law

$$\frac{\sin\alpha_i}{\sin\alpha_o} = \frac{n_o}{n_i} \tag{2.55}$$

with α_i and α_o the angles of the 'incoming' and 'outgoing' light with the normal to the interface at the point of intersection and n_i and n_o the refraction indices at the 'incoming' and 'outgoing' side of the interface, respectively. See Figure 2.41.

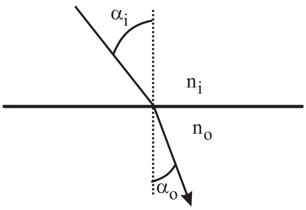

Figure 2.41 Refraction of a light beam at an interface separating two layers of different index of refraction.

Intermezzo: Snell's law

Snell's law:

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{n_2}{n_1} \tag{2.56}$$

was first formulated by Willibrord Snellius (a Dutch scientist) in 1621. However, Snellius did not publish his findings. That was much later done (in 1703) by the famous Dutch scientist Christiaan Huygens, who credited it to Snellius. Snellius found his law from observations.

Somewhat later, the French scientist Pierre de Fermat (1601-1665) arrived at the same law based on the physical argument that the light will travel along the path requiring minimum time. Consider Figure 2.42: the light starts at point $P = (p_x, p_y)$ in medium 1, where the speed of light is v_1. It crosses the interface between medium 1 and 2 at point $M = (m_x, 0)$ and passes point $Q = (q_x, q_y)$ in medium 2, where the speed of light is $v_2 \neq v_1$. As light moves in a straight line in a given, homogeneous medium, the time taken to move from P to M to Q is:

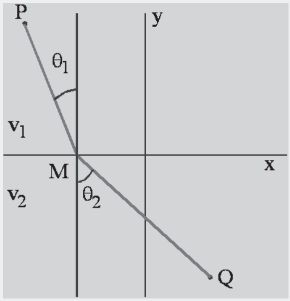

Figure 2.42 Light path from P to Q.

$$t = t_{PM} + t_{MQ} = \frac{PM}{v_1} + \frac{MQ}{v_2} = \frac{\sqrt{(m_x - p_x)^2 + p_y^2}}{v_1} + \frac{\sqrt{(m_x - q_x)^2 + q_y^2}}{v_2} \tag{2.57}$$

According to Fermat, the position of M is such that t is a minimum. Thus,we differentiate t with respect to m_x and set the derivative to zero:

$$\frac{dt}{dm_x} = \frac{m_x - p_x}{PM \cdot v_1} + \frac{m_x - q_x}{MQ \cdot v_2} = 0 \rightarrow \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \tag{2.58}$$

If we now use the definition of the refraction index: $n_i = \frac{c}{v_i} > 1$, we indeed find Snell's law.

As the refraction index of the atmosphere will change continuously, we will have to set up a differential equation that describes the changes of the angles in Snell's law from place to place. This is complicated by the fact that the atmosphere is folded around the earth, hence its geometry is curved as well. Let's for simplicity assume, that the atmosphere consists of concentric layers, each with a thickness Δr and an index of refraction $n(r)$. A light beam will 'hit' the outer surface at position r at an angle α_i and move on with an angle α_o. These angles are defined with respect to the normal at the interface at position r. As the beam follows its path, it will 'hit' the inner surface of the same layer, which is located at position $r - \Delta r$. Even though the path of the beam in this layer is a straight line, it will not 'hit' the interface at $r-\Delta r$ with the same angle α_o as it has left the interface at r. This is caused by the curvature of the layer. In a flat geometry this would be the case and the analysis would be relatively easy. The two cases are clarified in Figure 2.43.

Example of Snell's law

Snell's law not only applies to light, but to other phenomena where motion in two different media is found as well. As a simple example, consider the following situation. You are a bay watch and spot a swimmer at sea in trouble. The swimmer is out at sea a given distance to your right. Obviously, it is your task to reach the swimmer as soon as possible. Being a bay watch, you are a great athlete and able to run and swim fast. Nevertheless, running goes at a speed v_r which is much faster than the swimming speed v_s. What do you need to do? The answer is now simple: follow the path of least time and, hence, chose the one that obeys Snell's law.

Of course, being a bay watch you don't have the time to figure out what exactly the running & swimming track is according to Snell. But the second best option is realizing that you run much faster than you can swim; $\frac{v_s}{v_r} << 1$. Hence in Snell's law:

$$\frac{\sin\theta_r}{v_r} = \frac{\sin\theta_s}{v_s} \rightarrow \sin\theta_s = \sin\theta_r \frac{v_s}{v_r} \approx 0 \tag{2.59}$$

In other words, you run in a straight line until you are exactly at the same height of the swimmer and swim the shortest distance. This is thus not the shortest time, but it will be very close as any realistic example will show.

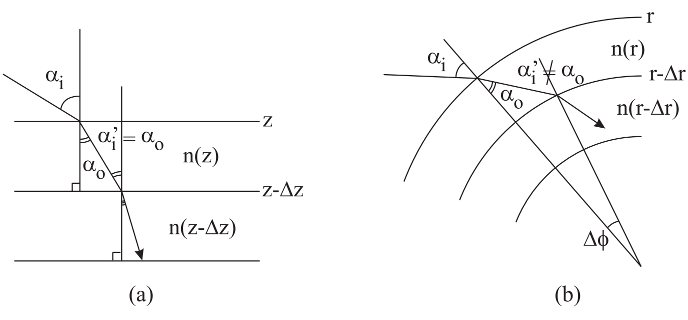

Figure 2.43 Path of light through a medium with a changing index of refraction: (a) flat geometry, (b) curved geometry.

For the curved situation we still have Snell's law at each interface, but now $\alpha_o \neq \alpha_i'$. The light beam intersects the inner surface at a position where the normal is rotated over a small angle compared to the normal at the intersection point with the outer surface. The relation between the two angles is found from Figure 2.46:

$$\alpha_i' = \alpha_o + \Delta\phi \tag{2.60}$$

We can use that Δr is small, thus $\Delta\phi$ is small and thus the curvature of the line segment l can be neglected. Therefore, it holds that:

$$\begin{aligned} \tan\Delta\phi &= \tfrac{l}{r-\Delta r} \\ \tan\alpha_o &= \tfrac{l}{\Delta r} \end{aligned} \qquad (2.61)$$

Thus, eliminating l we have a relation between α_o and $\Delta\phi$:

$$\frac{\tan\Delta\phi}{\tan\alpha_o} = \frac{\Delta r}{r-\Delta r} = \frac{\Delta r}{r} + h.o.t. \rightarrow$$
$$\Delta\phi = \tan\alpha_o \cdot \frac{\Delta r}{r} \qquad (2.62)$$

Intermezzo: Willebrord Snellius

Willebrord Snell, or Snellius in its Latin form, was born in Leiden, The Netherlands, in 1580. His father was professor of mathematics at Leiden university, but Willebrord decided to enter Law School. However, he switched to mathematics. In 1613 he succeeded his father as professor of mathematics at the University of Leiden. In 1617 Snell published his methods for measuring the Earth.

Figure 2.44 Willebrord Snellius (1580-1626)

He proposed the method of triangulation, which is the foundation of geodesy. Snell discovered his law of refraction in 1621. However, he did not publish it. Thanks to Christiaan Huygens Snell's name is attached to this law. After Huygens' death (1695) in 1703 Huygens' famous work *Dioptrica* was published in which he mentions Snell's law and attributes it rightfully to Snellius.

Figure 2.45 Christiaan Huygens (1629-1695)

So, in principle, we now have connected α_i to α_o (via Snell's law) and α_o to α_i' and Δr. At this stage it is important to realise that $\alpha_i' = \alpha_i(r-\Delta r)$, in words: α_i' is the angle that the incoming angle α_i has at layer position $r-\Delta r$. Thus, we can write eq.(2.60), using the above equation, as:

$$\alpha_i(r-\Delta r) = \alpha_o + \tan\alpha_o \frac{\Delta r}{r} \qquad (2.63)$$

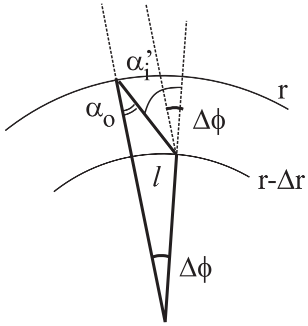

Figure 2.46 The two normals make a small angle due to the curved geometry.

We can expand: $\alpha_i(r-\Delta r) = \alpha_i(r) - \frac{d\alpha_i}{dr}\Delta r$ and take the sinus of both sides of the above equation:

$$\sin\left(\alpha_i - \frac{d\alpha_i}{dr}\Delta r\right) = \sin\left(\alpha_o + \tan\alpha_o \frac{\Delta r}{r}\right) \tag{2.64}$$

Remember that this equation gives us the change of the direction of the light beam. If we use the summation rule for the sinus $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$, we can write:

$$\begin{aligned} \sin\left(\alpha_i - \frac{d\alpha_i}{dr}\Delta r\right) &= \sin(\alpha_i)\cos\left(\frac{d\alpha_i}{dr}\Delta r\right) - \cos(\alpha_i)\sin\left(\frac{d\alpha_i}{dr}\Delta r\right) = \\ &= \sin(\alpha_o)\cos\left(\tan\alpha_o\frac{\Delta r}{r}\right) + \\ &\quad \cos(\alpha_o)\sin\left(\tan\alpha_o\frac{\Delta r}{r}\right) \end{aligned} \tag{2.65}$$

Next, we will simplify this equation by using that Δr is small. For small x we have $\sin(x) = x$ and $\cos(x) = 1$. Thus, the above equation can be simplified to:

$$\begin{aligned} \sin(\alpha_i) - \cos(\alpha_i)\left(\frac{d\alpha_i}{dr}\Delta r\right) &= \sin(\alpha_o) + \cos(\alpha_o)\left(\tan\alpha_o\frac{\Delta r}{r}\right) \\ &= \sin(\alpha_o) + \sin(\alpha_o)\frac{\Delta r}{r} \end{aligned} \tag{2.66}$$

We can eliminate α_o using Snell's law:

$$\begin{aligned} \sin\alpha_o = \frac{n_i}{n_o}\sin\alpha_i &= \frac{n(r)}{n(r-\Delta r)}\sin\alpha_i \\ &= \frac{n(r)}{n(r) - \frac{dn}{dr}\Delta r}\sin\alpha_i \\ &= \left(1 + \frac{1}{n}\frac{dn}{dr}\Delta r\right)\sin\alpha_i \end{aligned} \tag{2.67}$$

Finally, if we now combine eq.(2.66) and eq.(2.67) we have the differential equation that describes the evolution of the angle α_i of the light beam we were looking for:

$$-\cos\alpha_i \cdot \frac{d\alpha_i}{dr} = \left(\frac{1}{n}\frac{dn}{dr} + \frac{1}{r}\right)\sin\alpha_i \tag{2.68}$$

The above equation can easily be solved. We use as boundary condition that at the edge of the atmosphere the refraction index equals 1 and the incoming angle depends on what light beam we are trying to trace. It may be from the sun or any star. So, we have at $r = R + H$ (R denotes the earth radius and H the thickness of the atmosphere). $n = 1$ and we will call the incoming angle α_{in}. The solution then is:

$$\frac{\sin\alpha_i}{\sin\alpha_{in}} = \frac{1}{n(r)}\frac{R+H}{r} \tag{2.69}$$

Although we now know the angle a light beam makes with each normal on its way from space to the observer (at e.g. ground level), we still haven't solved everything. We still don't know the direction of all those normals, as they depend on where the light beam is in the atmosphere.

What needs to be done is to reconstruct the entire beam path and look for that particular beam from the sun, whose path will be such that it reaches the observer. As the sun is far away from the earth, half of the atmosphere is always hit by the sun's light beams. These are, due to the large distance from sun to earth, virtually parallel to one another. However, their incoming angle changes with the position at which they enter the atmosphere, as the normal at the point of incidence changes from point to point.

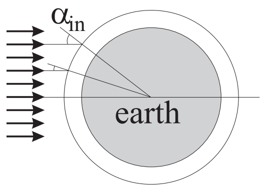

Figure 2.47 Different angles for incoming sunlight

The light path can best be desribed in cylinder coordinates $\{r, \phi\}$, where we define $\phi = 0$ as the direction of the vertical (= zenith). Further, our observer is at ground level and thus has coordinates $\{R,0\}$, see Figure 2.48. The relation between ϕ and r is already given in eq.(2.62):

$$\frac{d\phi}{dr} = \frac{\tan\alpha_o}{r} \tag{2.70}$$

Furthermore, α_o is coupled to α_i via Snell's law:

$$\frac{\sin\alpha_i}{\sin\alpha_o} = \frac{n(r-\Delta r)}{n(r)} = 1 - \frac{1}{n}\frac{dn}{dr}\Delta r \tag{2.71}$$

Thus we have: $\alpha_o = \alpha_i + \mathscr{O}(\Delta r)$. Putting this into eq.(2.70) gives the relation between ϕ, r and $\alpha_i(r)$. The latter is a known function of r and $n(r)$, i.e. eq.(2.69).

Putting it all together gives:

$$\begin{aligned}\frac{d\phi}{dr} &= \frac{\tan\alpha_i}{r} \\ &= \frac{1}{r}\frac{\sin\alpha_i}{\sqrt{1-\sin^2\alpha_i}} \\ &= \frac{R+H}{nr^2}\frac{\sin\alpha_{in}}{\sqrt{1-\sin^2\alpha_{in}\left(\frac{R+H}{nr}\right)^2}}\end{aligned} \tag{2.72}$$

The last equation does look complicated, but it can elegantly be simplified by introducing the coordinate $s \equiv \frac{r}{\sin\alpha_{in}(R+H)}$:

$$\frac{d\phi}{ds} = \frac{1}{s\sqrt{s^2n^2-1}} \tag{2.73}$$

This equation describes the trajectory of a light beam through the atmosphere. We have arrived at a good point to test whether or not our description makes any sense. We will analyze the situation in which the refraction index of the atmosphere is 1, i.e. what an observer would experience on the moon. Furthermore, we will assume that the light beams enter the 'atmosphere' parallel to the horizon. Obviously, we expect that a light beam will continue on a straight line, as there is nothing to deflect it from its path. This situation is drawn in Figure 2.48.

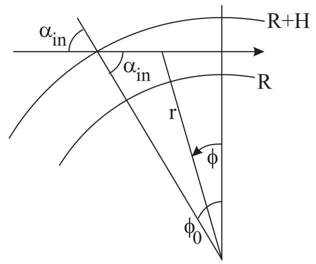

Figure 2.48 Straight path of light through an atmosphere with a refraction index of 1.

From simple geometrical considerations we see that the straight line of the beam is described by:

$$r = \frac{(R+H)\sin\alpha_{in}}{\cos\phi} = \frac{(R+H)\cos\phi_0}{\cos\phi} \tag{2.74}$$

Let's see whether eq.(2.73) gives the same result, when taking $n(r) = 1$. Thus, we have to solve

$$\frac{d\phi}{ds} = \frac{1}{s\sqrt{s^2-1}} \tag{2.75}$$

The general solution of this equation is:

$$\phi = \arccos\frac{1}{s} + Const \rightarrow \frac{1}{s} = \cos(\phi - C) \tag{2.76}$$

For the boundary condition, we have another look at Figure 2.48. The beam enters at $\{r_{in} = R+H, \phi_0\}$, making an angle α_{in} with the normal at that point. From Figure 2.48 it is clear that $\sin\alpha_{in} = \cos\phi_0$. Thus we have that $s_0 \equiv \frac{r_{in}}{\sin\alpha_{in}(R+H)} = \frac{1}{\sin\alpha_{in}} = \frac{1}{\cos\phi_0}$. If we put this boundary condition into eq.(2.76) it is found that the integration constant C equals $2k\pi$. Thus the solution for $s(\phi)$ is:

$$s = \frac{1}{\cos\phi} \rightarrow r = \frac{(R+H)\sin\alpha_{in}}{\cos\phi} \tag{2.77}$$

which is exactly the same result as we have found above from the geometrical consideration (eq.(2.74))!

We now can turn to solving the path of a light beam in the atmosphere of the earth. For simplicity we assume that the refractive index of the atmosphere changes linearly from 1 at the outer edge to 1.0003 at ground level: $n(r) = 1 + 0.0003 \cdot \frac{R+H-r}{H}$. It is easy to see, that light rays coming directly out of the zenith are not bend but continue on a straight line. We expect the biggest changes when light enters close to the horizon. Thus, we will analyse what the apparent position of the sun is when the upper edge of the sun just touches the horizon; the sun has just set. That is to say, it would do so if there was no atmosphere. We now ask the question: which of all the sun beams that come in parallel to the horizon and enter the atmosphere over its entire height is seen by the observer and under what angle with the horizon? From a numerical solution of eq.(2.73) we find that this angle is 0.27°. Hence, the observer still sees a large part of the sun above the horizon!

In reality, the apparent 'lift' of the sun's position close to the horizon is about 39 arc minutes. As the diameter of the sun is only 30 arc minutes, the sun is actually completely below the horizon when we see it touching it with its lower edge. This also means that the apparent motion of the sun slows down when the sun approaches the horizon. This is a simple way of showing that the earth is a curved object rather than a flat one, as for flat geometries the apparent motion differs from the curved ones.

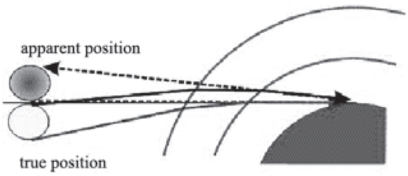

Figure 2.49 Refraction of light rays makes the sun appear above the horizon after setting.

Furthermore, the bending of the light paths is stronger for beams that enter the atmosphere closer to the horizon than those higher up. Consequently, the shape of the sun or moon gets distorted when they approach the horizon. The lower edge gets lifted up more than the upper edge: the sun (or moon) start to look ellipsoidal. As mentioned, normally the sun's diameter is 30', but close to the horizon it looks like its vertical dimension has shrunken to 24'. The sun's portrait is no longer a circle but it has become ellipsoidal.

There is of course another reason why the sun has already set, when we still see it above the horizon: the finite speed of light. Although light travels at an enormous speed, $c = 2.998 \cdot 10^8$m/s in vacuum, it takes a finite time for sunlight to reach the earth. As the average distance between the sun and earth is $150 \cdot 10^6$km, it takes 500s for light to reach us. This means that by the time this light has reached the earth, the earth has rotated over an angle of 2°.

2.7 The rainbow across the sky

Rainbows are a good example of the interplay between light and water. Water droplets in the atmosphere refract light. As the refractive index is a (weak) function of the wave length of light, different colors get refracted over different angles. The rainbow is only seen with the sun in the back. This is so, because the rain drops in the atmosphere internally refract and reflect the light back and send it back. The paths of light are shown in Figure 2.50. From this the primary rainbow is formed.

primary rainbow

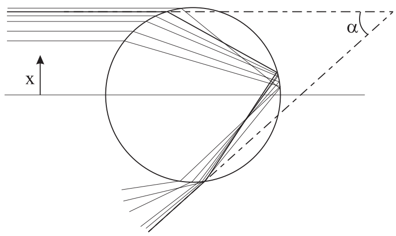

Figure 2.50 Light rays (of a single wave length) refracted through a water droplet.

There are also rainbows in the direction of the sun. However, they are rather weak and much weaker than the sunlight itself.

light path through a droplet

It is easy to show that for a spherical droplet the maximum angle between the incoming and outgoing light is 42°. Consider Figure 2.51. Light coming in, is making an angle α_{in} with the normal on the droplet surface. According to Snell's law the light beam is refracted towards the droplets normal, making an angle α_{out}. It hits the droplet surface from the inside at an angle ϕ with the normal. Part of the light is reflected back into the droplet, obviously also at an angle ϕ with the normal. Then it reaches the surface again, now at an angle α'_{in} with the normal and is refracted out of the droplet with an angle α'_{out}. The angle describing the total deviation is called α.

From Figure 2.51 it is clear, that the two triangles, IMP and OMP, within the droplet are identical, hence $\alpha_{out} = \alpha'_{in}$. Consequently, $\alpha_{in} = \alpha'_{out}$. Furthermore, $\phi = \alpha_{out}$ as the sides IM and MP in the triangle IMP are equal. So, for the total deviation of the light beam path we find:

$$\begin{aligned} \alpha &= 4\alpha_{out} - 2\alpha_{in} \\ &= 4\arcsin\left(\frac{\sin\alpha_{in}}{n}\right) - 2\alpha_{in} \end{aligned} \tag{2.78}$$

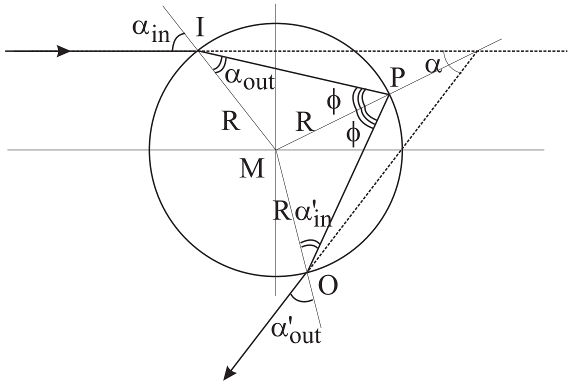

Figure 2.51 Geometry of a refracted light beam through a water droplet.

The incoming angle, α_{in}, varies from 0 to $\frac{\pi}{2}$. Thus, we can calculate the variation of the deviation, α. It shows a maximum at $42.01°$ for $n = 1.3335$. This is the value for the refraction index for (red) light with a wave length of 600nm. In Figure 2.52 the deviation angle is shown.

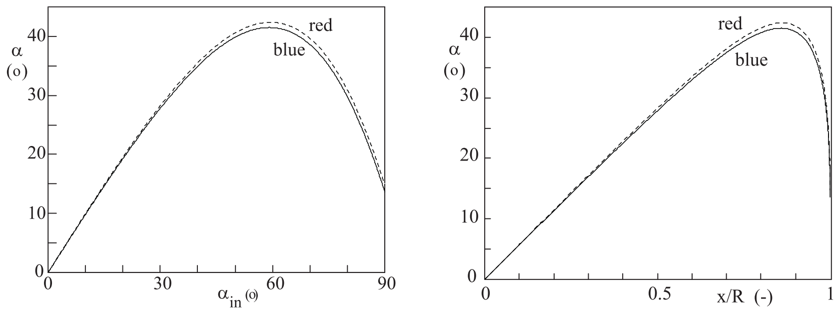

Figure 2.52 Deviation of light beam that is refracted and reflected by a spherical water droplet: (a) as a function of the incoming angle α_{in}, (b) as a function of x/R.

The incoming light hits the droplet at various positions. We denote this by the coordinate x, see Figure 2.50. As figure 2.52b shows, there is region for x/R between say 0.8 and 0.9 where all the incoming light is 'focused' in a small region of the deviation angle. This obviously means that this portion of light is seen with a much greater intensity by the observer. This light forms the rainbow.

The maximum deviation angle depends on the refraction index, which in turn is a (weak) function of the wave length. This dependence is plotted in Figure 2.53.

Thus, we see that the colors of the rainbow run from red at the outside to blue at the inside. Notice further, that the separation between the colors blue and red is $1.5°$. Considering that the effective diameter of the sun is less than $0.5°$ we find that this separation is sufficient to have a rainbow with separate colors, a fact we obviously knew.

colors of the rainbow

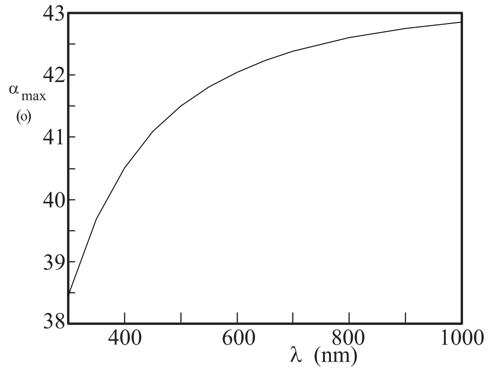

Figure 2.53 Maximum deviation against wave length.

2.7.1 Secondary rainbow

The rainbow discussed above is caused by light that is reflected once inside the droplet. There is, however, the possibility of multiple internal reflections. These generate different deviation angles and thus more rainbows can be formed simultaneously. Under the right circumstances, a secondary rainbow can be seen.

light path for secondary rainbow

The light path through a droplet for a secondary rainbow is shown in Figure 2.54. Now there are two internal reflections and of course the beam is refracted twice. As shown in the figure, all 'internal' angles are equal. The deviation angle, α_s, is related to the incoming angle, α_{in}:

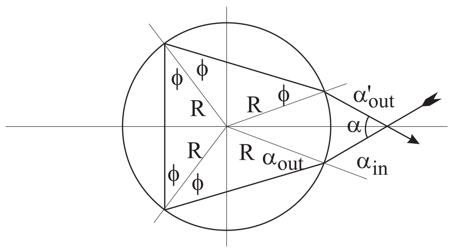

Figure 2.54 Geometry of a refracted light beam that forms the secondary rainbow.

$$\alpha_s = \pi - 6\arcsin\left(\frac{\sin\alpha_{in}}{n}\right) + 2\alpha_{in} \tag{2.79}$$

This is plotted in Figure 2.55, together with the result we had for the primary rainbow both for a wave length of 600nm.
There are a few matters to be noticed in this figure.

- The deviation shows a minimum, rather than a maximum, at $\alpha = 51°$.
- There is a 'forbidden band' in the deviation angles from about 42° to 51°.

Alexander's band

The second point means that there is a dark band between the primary and secondary rainbow, where no light refracted by droplets is sent into the direction of the observer. This is known as Alexander's band , in honor of the Greek Alexander of Aphrodisias, who first pointed out that the area between the two red bands of the primary and secondary bows is darkened rather than brightened. In Figure 2.56 the relation of the deviation angle with the wave length of the light is plotted for both the primary and the secondary rainbow.

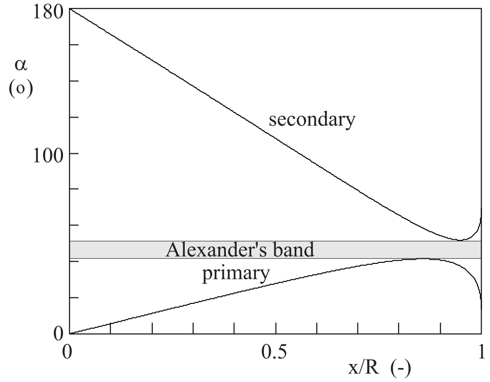

Figure 2.55 Deviation of light beam that is refracted and reflected by a spherical water droplet.

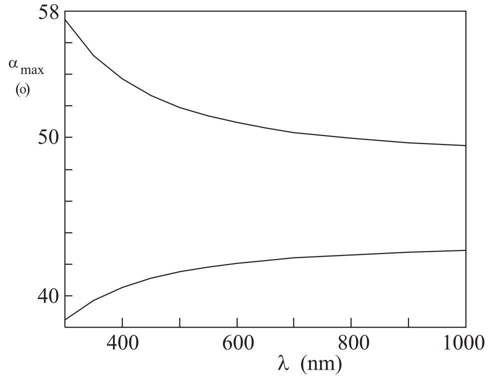

Figure 2.56 Maximum deviation against wave length.

colors of secondary rainbow

We see, that the secondary rainbow is inverted with respect to the primary one, now the colors run from red at the center towards blue at the outside (see Figure 2.57).

higher order rainbows

Of course, the intensity of the primary rainbow is greater than that of the secondary. This has two main reasons. in the first place, the light intensity decreases after each (internal) reflection. Secondly, the secondary rainbow is broader than the primary one: the 'focusing' of light along the minimum. Tertiary and higher order rainbows are present each with one more internal reflection than the previous one. Obviously, each higher order rainbow becomes fainter and broader and is therefore less visible. Very few reliable reports exist about the observations of the tertiary and quaternary rainbows. Even higher order rainbows have not been observed. In table (2.2) for the first 5 orders the mean angle, width and intensity relative to the primary are listed.

order	angle (°)	width (°)	rel. intensity
1	42.4	1.72	1
2	50.4	3.11	0.43
3	137.5	4.37	0.24
4	137.2	5.58	0.15
5	52.9	6.78	0.10

Table 2.2 Primary and higher order rainbows

Figure 2.57 Primary and secondary rainbow.

2.7.2 What about tertiary rainbows?

There are reports from observations of the tertiary and even quartiary rainbow. Are these all fake?

The answer is: yes and no. Yes, because as we see from table (2.2) the third and fourth rainbow are located in the direction of the sun and can not be seen by the naked eye. And no, there are special circumstances where a third rainbow is seen. See for example Figure 2.58.
How can that be? What is this mysterious third rainbow. The answer is that it is the reflection of the part of the primary rainbow that can not be seen, in a lake that is in front of the photo. Hence, it is not the tertiary rainbow, but rather a part of the primary one. This is clarified in Figure 2.59.

2.8 Soap bubbles, butterflies and light

soap bubble

The colors of the rainbow also show up when looking at soap bubbles, beit not in a particular order. Here, the mechanism causing this is not refraction, but 'double' reflection.
The light that shines on the bubble is reflected twice, once on the outside of the soap film of the bubble, once on the inside of the soap film. The soap film is very

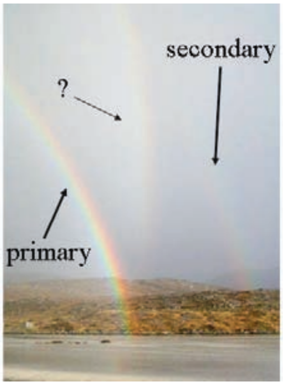

Figure 2.58 Primary, secondary and tertiary rainbow?

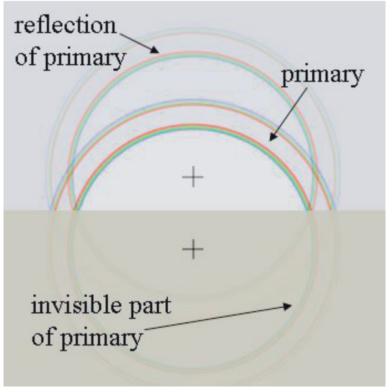

Figure 2.59 Third rainbow coming from a reflection on water.

thin, on the order of a micro-meter. This causes the two reflected light 'beams' to interfere. Light can be considered as waves. These waves are not infinitely long, but form small packages that are independent from each other. This is schematically shown in Figure 2.61.

wave package

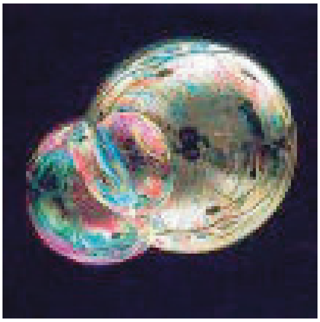

Figure 2.60 Colorful soap bubble.

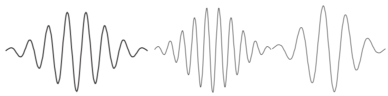

Figure 2.61 Light as wave packages.

After the waves have reflected from the soap film, they interfere and form new wave packages. If the film thickness is smaller than the size of the packages, the waves add if they are in phase and they cancel each other out if they are out of phase.

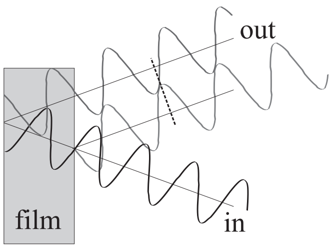

Figure 2.62 Interfering waves after reflection.

In Figure 2.62 an example is given. Here the two reflected waves are almost in opposite phase (see the wave amplitude at the dashed line). Consequently, these waves will cancel each other out almost completely and no reflected light will be seen (or only very weak with a small amplitude). If, on the other hand the two waves move in phase, they will add up to give the reflection a larger amplitude. Whether there is amplification or canceling out of the two reflected waves depends on the distance between the reflecting layers, the angle of reflection and, of course, on the wave length of the wave. For complete destruction the two reflected waves should be shifted with respect to each other by exactly half a wave length.

Figure 2.63 Colorful wings.

Figure 2.64 More colorful wings.

Different colors are from a physical point of view waves with different wavelength. As we have seen before, different colors will have different amplitudes after the double reflection. Depending on the thickness of the soap film, some colors will be reflected at maximum intensity, others will almost completely disappear. That is why we see at different spots different colors and the colorful patterns develop.

This only works if the incoming wave is coherent over a 'long distance'. We mean by this, that the package is so long, that the first reflection and the second reflection when meeting each other actually come from the same package. If this is not the case, they may have very different phase and the outcome of the addition can be anything. For our eye this means that the total light of a particular color that comes in will be a mix of 'weakening and addition'. This will be fluctuating and we will not observe anything particular. If, however, the soap film is thin, the interference will work well for each wave package; the reflected waves originate from the same packages and have started with the same phase. Thus, thin soap films can generate the colorful display, thick ones can't. This not only holds for soap films. Various animals use the same trick. For instance, the colorful wings of a butterfly have a top layer that generates two reflections. This layer is very thin and varies in such a way over the wing that a colorful pattern becomes visible. All that is needed is to have that pattern in the thickness of the layer distributed in the right way over the wing.

CD ROM

There is another 'trick' that is used with interference of light. Sharp contrast can be reached by making use of complete demping. This can be reached by making a step change in the depth of the reflecting surface. Our CD ROM technology employs this.

CD ROM

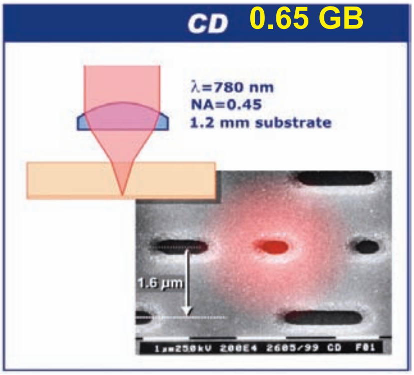

Figure 2.65 The red laser light trying to read a CD ROM.

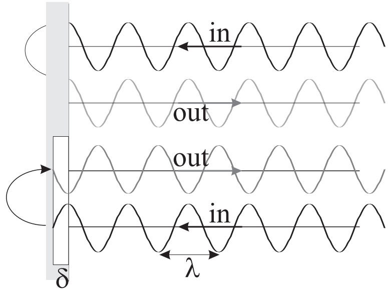

Figure 2.66 Destructive interference using the quart lambda trick.

The information on a CD ROM is written in the form of small dimples. Laser light is focused on the CD ROM and the reflections are recorded. A dimple gives a different intensity than the flat surface. To sharpen this effect, it is desirable to give the red dot of the laser light hitting the dimple dark edges. How can we do that? The laser light will reflect anywhere from the CD surface. Here, interference of the laser light is used.

The depth of the dimple is one quarter of the value of the wave length, λ, of the laser light used. This means that light reflected just inside the dimple has to travel a path that is 0.5λ longer than the light that is reflected just outside the dimple. As a consequence, the two waves will be opposite in phase and they will cancel each other out, see Figure 2.66. Thus, the detector will not receive any light from and around the edge of a dimple. This makes discrimination of the dimples much better. Obviously, this contributes to error free reading of the CD ROM.