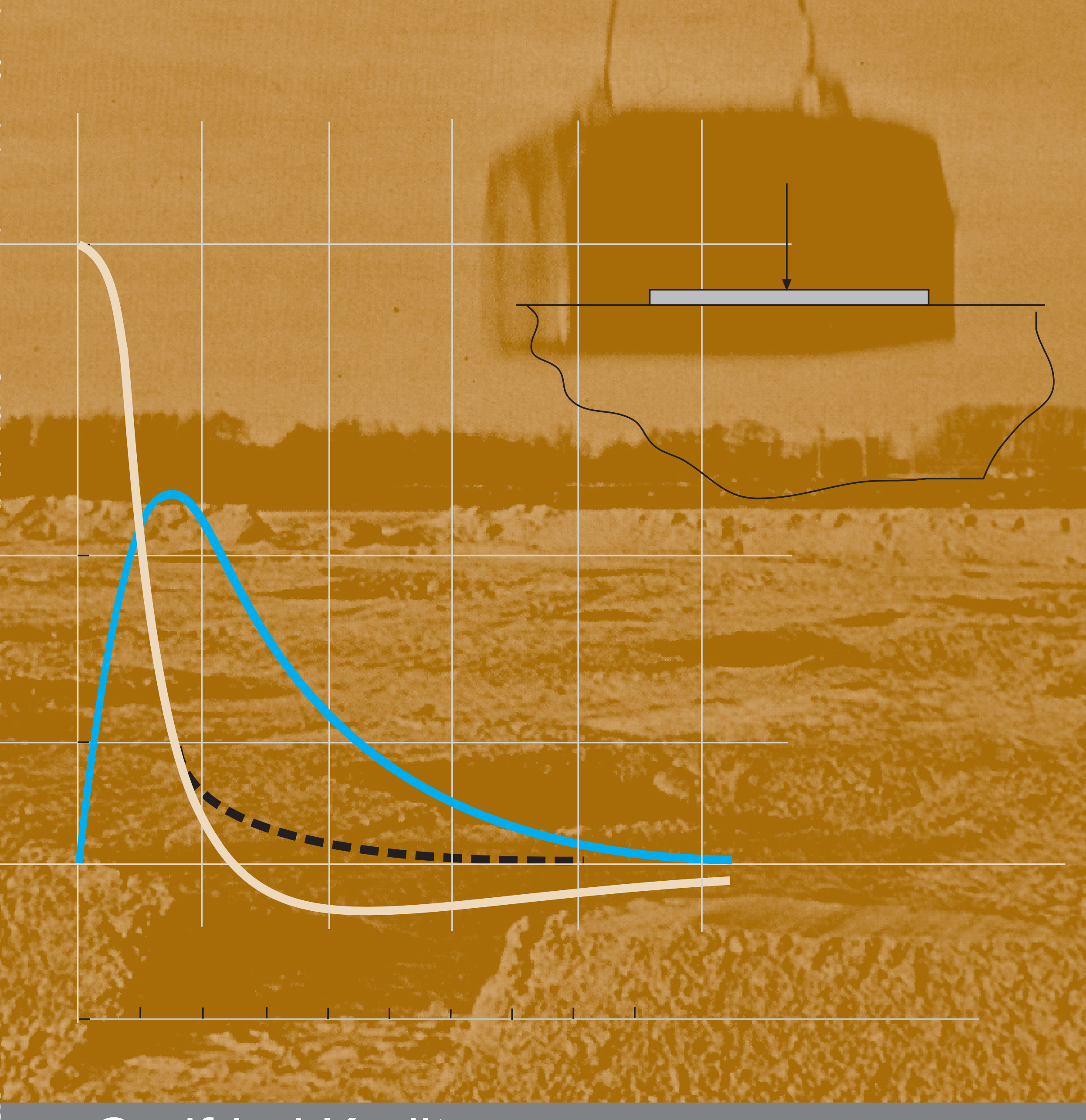
The vertical motion of foundations and pontoons



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Preface

This text collects the papers 'Vertical Vibration of Rigid Bodies on Deep Elastic Strata' and 'A Stoneley-Gibson-Varga Elastic Stratum' that have been published in the journal Heron (Volume 46, no. 1 (2001)).

The first chapter offers a survey of the vertical motion of rigid bodies resting on deep elastic strata. Four strata are distinguished:

- deep water,
- the homogeneous isotropic elastic half-space,
- the water saturated homogeneous isotropic porous elastic half-space and
- the Gibson half-space.

Four types of footings are considered: the strip, the circular disk and the embedded semi-cylinder and hemi-sphere.
In particular attention has been given to the distinction between compressible and incompressible strata, and to the distinction between low and high frequency factors of the oscillatory motion.

The second chapter provides a geometrically non-linear generalization of the Gibson soil. Some remarkable solutions concerning excavations and indented rigid punches are presented.
The results provide a first approximation of the behaviour of foundations on real soils in the case of small soil strains.

The author is indebted to Prof dr Ton Vrouwenvelder of Delft University of Technology and to Jacques Schievink, publisher of text books at VSSD. They provided indispensible encouragement, helpful criticism, cooperation and assistance in preparing the final text and illustrations.

Voorburg, November 2002

Contents

1. A comparative treatise on the vertical vibration of rigid bodies on deep elastic strata

Deep water, Gibson soil, Homogeneous (water saturated porous) isotropic elastic half-space
Compressible versus incompressible strata
Low frequency versus high frequency factors

Keywords: **lumped parameters, dynamic subgrade reaction, floating bodies, resting footings.**

1.1. Introduction

In soil mechanics a Gibson soil is defined as being an incompressible, isotropic, elastic half-space $X_1 \geq 0$ in which the shear modulus μ increases linearly with depth X_1 from zero on the upper surface $X_1 = 0$, according to the equation $\mu = mX_1$ with m a positive constant.
In his famous 1967-paper Gibson showed that the upper surface of this elastic deep stratum reacts under static normal loading like a uniform bed of springs, a so-called Winkler foundation (Gibson (1967)).
Any point of the upper-surface $X_1 = 0$ settles an amount w $(X_1 = 0)$, directly proportional to the local intensity $-q(X_1 = 0)$ of applied normal stress according to the law $w(X_1 = 0) = q(X_1 = 0)/(2m)$; outside the loaded area the upper-surface does not settle.
It has been noticed that 1) the induced deformation at the locations $X_1 > 0$ is irrotational, 2) the state of stress on the location of the (loaded) upper surface $X_1 = 0$ is purely isotropic and 3) the settlement $w(X_1)$ of a point at the level $X_1 > 0$ is directly proportional to the all-round pressure $-p(X_1)$ at that point according to $w(X_1) = p(X_1)/(2m)$. These settlements at the levels $X_1 > 0$ decrease with increasing horizontal or vertical distances from the loaded surface area. Of course, the state of stress on the planes $X_1 > 0$ is not isotropic since the Gibson soil possesses shear rigidity at planes $X_1 > 0$. Further, it has been realized (Lekhnitski (1962) and Gibson (1967)) that the stress distribution in the Gibson soil due to a static normal surface loading corresponds exactly to the stress distribution in an incompressible homogeneous isotropic elastic half-space due to the same surface loading.
When it is assumed that the Gibson soil has been subjected initially to an hydrostatic stress distribution due to its self-weight own weight, the quantity $2m$ must be replaced by $(\rho g$+$2m)$ with ρ the uniform mass density and g the acceleration due to gravity.
It has been shown that the linear equations governing the dynamics of the Gibson soil resemble mathematically the linearized equations of the deep water motion (Kruijtzer (1976)). For example, when we analyse a group of plane harmonic waves of wave length $\lambda/2\pi\kappa$ travelling through the Gibson soil in a horizontal direction with velocity c, it is found that there exists irrotanional surface waves with wave velocity $c = ((\rho g + 2m)/(\rho\kappa))^{1/2}$. These irrotational waves are mathematically similar to the (irrotational) gravitational deep water surface waves with velocity $c = (g/\kappa)^{1/2}$.
In a recent paper we have introduced the geometrically non-linear Gibson soil, the so-called Stoneley-Gibson-Varga elastic half-space, in which the actual strees and the left stretch tensor are

correlated (Kruijtzer (2001)). It was shown that the non-linear equations of the irrotaional dynamics of this half-space resemble mathematically the classical non-linear equations of the irrotanional deep water motion. Furthermore, it was shown that the settlement $w(X_1)$ of a point at the level $X_1 > 0$ is directly proportional to an all-round pressure $-p(X_1)$ at that point according to $w(X_1) = p(X_1)/(\rho g + 2m)$. These settlements at the levels $X_1 > 0$ decrease with increasing horizontal or vertical distances from the loaded surface area. Of course, the state of stress on the planes $X_1 > 0$ is not isotropic since the Gibson soil possesses shear rigidity at planes $X_1 > 0$. It may be noticed that the static stress distribution in the Stoneley-Gibson-Varga elastic half-space is not similar to the stress distribution in the corresponding geometrically non-linear incompressible homogeneous elastic half-space.

In this treatise we compare the responses of deep water, the Gibson soil and the homogeneous isotropic elastic half-space on low and high frequency vertical surface loadings including the effects of compressibility and incompressibility of these strata. Our comparative treatise reveals not only various mathematical and physical resemblences or similarities,but also provides with an application in soil mchanics.

In theoretical soil mechanics water saturated soils are often conceived to behave like water saturated porous elastic strata (elastic skeletons). In the *fully drained* state there is no excess of water pressure so that the stratum behaves like an ordinary elastic medium. In the *fully undrained* state the water velocity equals the solid velocity. In this case the medium behaves as being an elastic medium but the modulus of compression of the medium depends mainly on the elasticity of water volume and scantly on the bulkmodulus of the elastic skeleton (ensemble of packed grains).

The fundamental 1956-paper of G. de Josselin de Jong is our guide in considering the corresponding responses of a water saturated porous isotropic elastic half-space.

Finally we notice that the truncated semi-infinite cone-model of the elastic half-space for vertical vibrations (J.P.Wolf (1985)) is based on the results of the classical half-space theory.

1.2. Motion of footings and floating bodies

Linear dynamics

We want to compare the vertical motion of a rigid footing resting on the surface of a Gibson soil or a homogeneous incompressible isotropic deep elastic stratum with the corresponding heave motion of a rigid floating body on deep water (Figure 1.1). We did not find non-linear solutions, so we restrict ourselves necessarily to solutions of the linearised equations of motion. In the linearised theories the difference between the gradients with respect to the material coordinates and the gradients with respect to the spatial coordinates is disregarded. Further, particular stress distributions due to special external loadings may be superposed.

In general, the behavior of linear systems can be described in either the time domain or the frequency domain. It may be well-known that the characteristics of the high frequency steady state motion of such systems due to external periodic loading may be obtained from the initial motion of these systems due to an impulsive external loading (Maskell and Ursell (1970) in Ursell (1994), Cummings (1962) and Ogilvie (1964)).

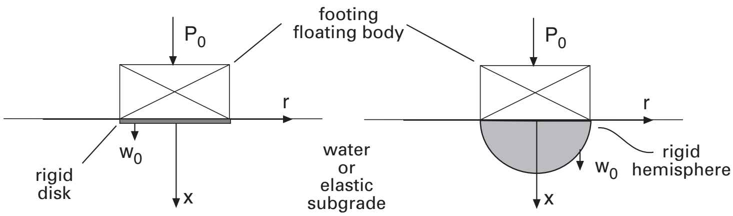

Figure 1.1. Footing on an elastic subgrade or floating body on deep water.

Single degree of freedom model

In engineering mechanics the vertical motion of footings and floating bodies is often described by a lumped-parameter-system with a single-degree-of-freedom. With $w_0 = w_0(t)$ the vertical displacement of the rigid body and $P_0 = P_0(t)$ the external vertical centric loading the equation of motion of the body reads

$$(M^{\circ} + \overline{M}^{*})\ddot{w}_0(t) + \overline{C}\dot{w}_0(t) + \overline{K}w(t) = P_0(t) \tag{1.2.1}$$

with M° the footing mass, $\overline{M}^{*}$ the added (in-phase) subgrade mass, $\overline{C}$ the coefficient of damping due to wave radiation through the subgrade, and $\overline{K}$ the coefficient of the subgrade restoring force.

The value of $\overline{M}^{*}$, $\overline{C}$ and $\overline{K}$ depend on the induced subgrade motion.

With $P_0(t) = \hat{P}_0 e^{i\omega t}$ and $w_0(t) = \hat{w}_0 e^{i\omega t}$ the steady state motion of the linear system is given by the equation

$$-(M^{\circ} + M^{*}(\omega))\omega^2 \hat{w}_0 + i\omega\, C(\omega)\hat{w}_0 + K(\omega)\hat{w}_0 = \hat{P}_0 \tag{1.2.2}$$

with $M^{*}(\omega)$ the added mass, $C(\omega)$ the damping factor and $K(\omega)$ the restoring coefficient. These quantities, $M^{*}(\omega)$, $C(\omega)$ and $K(\omega)$ are frequency dependent.

We consider the cases in which a weightless rigid circular disk or infinitely long rigid strip, an embedded rigid hemisphere or infinitely long semi-cylinder are attached to the upper surface of the subgrade (Figure 1.1) and are loaded by a vertical centric force $P_0 = P_0(t)$, so that the rigid bases undergo a vertical displacement $w_0 = w_0(t)$ according to

$$w_0(t) = F(t)\, P_0(t) \tag{1.2.3}$$

with $F(t)$ the response function. If $P_0(t) = \hat{P}_0\, e^{i\omega t}$ and $w_0(t) = \hat{w}_0\, e^{i\omega t}$ then the equation (1.2.3) of the steady state motion takes the form

$$\hat{w}_0\, e^{i\omega t} = -\,(f_1 + \mathrm{i} f_2)\; \hat{P}_0\, e^{i\omega t} \tag{1.2.4}$$

with f_1 and f_2 two functions of frequency being effectively the in-phase and out-phase components of the displacement $w(t)$. In Appendix 1.1 the results of Bycroft are schematically presented and completed.

Deep water

The incompressible inviscid fluid subgrade is assumed to be intially, at time $t = 0$, at a state of rest. We suppose that at time $t = 0_+$ the upper surface has been subjected to an impulsive pressure $-p_0(t = 0_+)$. According to the linearized theory this impulsive pressure is represented by the value $-\rho\varphi_0(t = 0_+)$, where φ_0 is the value of the velocity potential at the loaded area at time $t = 0_+$ (Stoker (1957), p. 150, Lamb (1945), p. 384):

$$p_0(r,t = 0_+) = -\rho\varphi_0(r,t = 0_+) \tag{1.2.5a}$$

with ρ the water mass density and r the radius of the loaded area. (We may notice that a permanent flow passing an obstacle generates at each instant of time an impulsive pressure on the obstacle).

We now consider the case in which a thin weightless rigid circular disk with radius r_0 at the upper surface is loaded by a vertical and centric impulsive force $\tilde{P}_0$. Then (Lamb (1945), p. 120 and p. 138)

$$\begin{aligned} p_0(r,t=0_+) &= -\rho\frac{2}{\pi}\cdot\ \dot{w}_0(t = 0_+)\cdot(r_0^2 - r^2)^{1/2} \quad (0 \le r < r_0) \\ &= 0 \qquad\qquad\qquad\qquad\qquad\qquad (r > r_0) \end{aligned} \tag{1.2.5b}$$

where $\dot{w}_0$ $(t = 0_+)$ is the initial velocity of the disk.

Integration of (1.2.5b) with respect to r gives

$$\tilde{P}_0 = \int_0^{r_0} p_0\ 2\pi r\ dr = \frac{4}{3}\ \rho r_0^3\ \dot{w}_0(t = 0_+) \tag{1.2.5c}$$

It follows that high frequency added mass $M^*$is given by $M^* = (4/3)\rho r_0^3$.

In the case of an infinitely long weightless rigid strip of width $2d_0$ the added mass M^* is given by $M^* = (1/2)\pi\rho d_0^2$ (Lamb (1945), p. 85).

In the theory of linearized ship motion the rigid disk is replaced by a semi-submerged sphere and the rigid strip is replaced by a semi-submerged infinitely long cylinder. The vertical impulsive force $\tilde{P}_0$ on a floating sphere of radius r_0 and with mass $M° = (2/3)\pi\rho r_0^3$ gives rise to the high frequency added mass $M^* = M°/2 = (1/3)\pi\rho r_0^3$. The vertical impulsive line-force on a floating cylinder of mass $M° = (1/2)\pi\rho r_0^2$ per unit length, gives rise to the high frequency added mass $M^* = M°$ (Ursell (1994)), Lamb (1945), pp. 80 and 124). These reults may be proved as follows.

We consider the translational motion of weightless rigid sphere in an infinite incompressible inviscid fluid. At each instant of time the moving sphere generates an instantaneous fluid motion. The fluid pressure contains a linear portion and a non-linear portion (Lamb (1945), p. 124). The non-linear portion generates a zero resulting force on the sphere. At the location of a plane through the center of the sphere perpendicular to the direction of motion of the sphere, the linear portion of the pressure vanishes while the non-linear portion does not vanish. The kinetic energy of the fluid in given by $(2/3)\pi\rho r_0^3\,\dot{w}_0^2$ with ρ the fluid mass density,r_0 the radius of the sphere and $\dot{w}_0$ the velocity of the sphere. The resultant effect of the fluid pressure in the direction of the motion is given by $-(2/3)\pi\rho r_0^3\,\ddot{w}_0$, so that $M^* = (2/3)\pi\rho r_0^3$ is the added in-phase mass. Since according to the linearised theory at the plane through the center of the sphere and pendicular in the direction of the motion of the sphere the pressure vanishes, we are lead into the following result.

When a rigid floating hemisphere of radius r_0 is subjected to a vertical centric impulsive force $\tilde{P}_0$ at time $t = 0$ (Figure 1.1) the sphere gets the downward initial velocity $w_0(t = 0_+)$ according to

$$\tilde{P}_0 = (M^\circ + M^*)\, w_0(t = 0_+) \tag{1.2.6}$$

with M° is the mass of the semi-sphere and M^* is the added in-phase fluid mass: $M^\circ = (2/3)\pi\rho r_0^3$ and $M^* = (1/3)\pi\rho r_0^3 = M^\circ/2$. In the case of an infinitely long circular semi-cylinder $M^\circ = M^* = (1/2)\pi\rho\, r_0^2$ (Lamb (1945), p. 77).

The analysis by Newman (1969) gives rise to a quantification of the high frequency added mass and to an indication of the low frequency added mass as follows.

The forced motion of a floating body resting on deep water generates surface waves due to gravity. On the free surface the pressure $-p_0$ is equal to zero, so that at this location the linearised condition $\omega^2\varphi - g\dot{w} = 0$ must be satisfied, with φ the velocity potential, $\dot{w}$ the vertical surface velocity and g the acceleration due to gravity (Lamb (1945), p. 363).

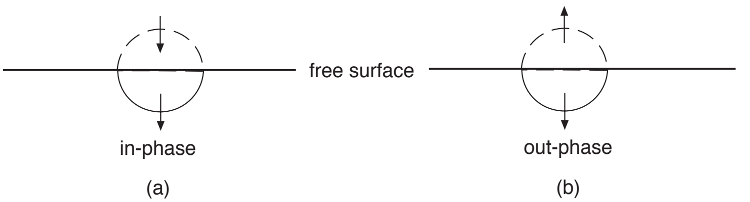

Figure 1.2. Method of images.

In the case of very high frequencies $\omega^2\varphi >> g\dot{w}$, so that the potential φ vanishes on the free surface (short waves). By the 'method of images' (Figure 1.2a) an upper hemisphere can be added to the lower hemisphere. When the upper and lower hemisphere move in-phase, the normal velocities on the upper and lower hemisphere are opposite in sign, so that the potential φ is equal to zero on the 'free surface'. This corresponds indeed to the problem of a sphere moving in a fluid of infinite extent, so that the high frequency mass added to the hemisphere is equal to $M^* = (\pi/3)\rho r_0^3 = M^\circ/2$.

In the case of very low frequencies ω we have $\omega^2\varphi << g\dot{w}$ on the free surface so that the normal surface velocity vanishes (long waves). When an image hemisphere is added to the submerged hemisphere and the upper and lower hemisphere move 180° ot of phase (Figure 1.2b), the vertical velocity at the original free surface is zero. In fact there is no longer a free surface problem, but the problem of pulsation of a dilitating 'sphere' of changing volume. From the litterature on ship motions it appears that the low frequency added mass is equal to about (3/2) times the high frequency added mass (compare the purely radial expansion motion of a sphere in a infinite incompressible fluid in which case φ does not vanish at the 'free surface' and the added mass is equal to $4{\cdot}(3/2)(2/3)\pi\rho r_0^3$ (Lamb (1945), p. 122)., but is smaller than twice the high frequency mass (Appendix 1.3). It is noticed that in two-dimensional problems the low frequency added mass becomes mathematically infinite, because from the continuity of finite flux of fluid, oscillating back and forth, there is only the way out at infinity. In three dimensions this infinity does not occur because the fluid flux can distribute itself spatially in all three directions. On the

other hand, long waves generate the influence of the 'bottom' of 'deep water' so that an infinite added mass does not arise.
In Figure 1.3 we present a schematic indication of the added mass M^* and the damping coefficient C.

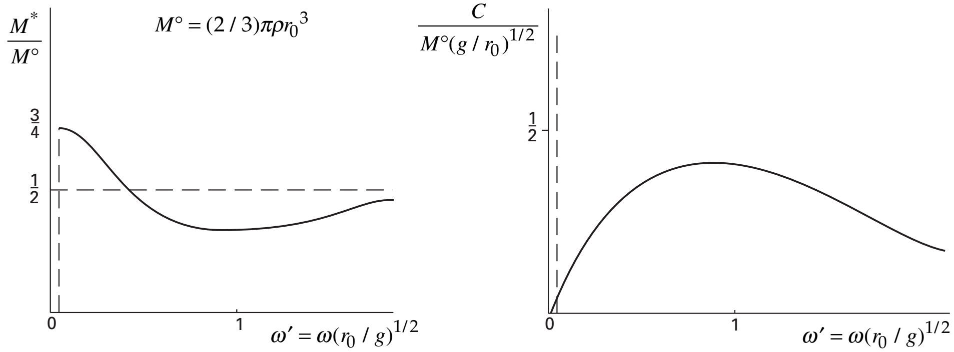

Figure 1.3. Schematic indication of the added mass M and coefficient of damping C.*

Let $\bar{A}$ be the frequency dependent 'wave-height ratio', i.e., the ratio between the amplitude w^* of the outgoing waves and the amplitude $\hat{w}_0$ of the oscillatory hemisphere:

$$\bar{A} = (w^*/\hat{w}_0) \text{ then } C = \rho\,(g/\omega)^2\,\bar{A}^{\,2}/\,\omega.$$

with $\bar{A} = \bar{A}(\omega)$ and $\bar{A}(0) = 0$.
We notice that it will appear that in the case of a rigid disk resting on a Gibson soil and an *incompressible* homogeneous elastic half-space comparable results concerning the low and high frequency added mass and damping coefficient arise.
Finally we notice that when a rigid weightless disk of radius r_0 rests on the upper surface of deep water and is subjected to a periodic displacement $w_0 = \hat{w}_0\, e^{i\omega t}$, the steady state stress distribution under the disk is given by the dual integral equations (compare Kruijtzer (1976) and appendix 1.2).

$$\bar{\hat{p}}_0(s) = \int_0^\infty \hat{p}_0(r) r J_0(rs) ds, \quad \hat{p}_0(r) = \int_0^\infty \bar{\hat{p}}_0(s) s J_0(sr) ds$$

and

$$\int_0^\infty \frac{\bar{\hat{p}}_0(s)\cdot s\cdot J_0(sr)}{\rho g - \dfrac{\omega^2\rho}{s}} ds = \hat{w}_0 \quad (0 \le r < r_0)$$

$$\hat{p}_0(r) = \int_0^\infty \bar{\hat{p}}_0(s)\cdot s\cdot J_0(sr)\,ds = 0 \quad (r \ge r_0) \tag{1.2.7a}$$

where we have used $w_0(t) = \hat{w}_0\, e^{i\omega t}$ and $p_0(r,t) = \hat{p}_0(r) e^{i\omega t}$. Further, $J_0(\alpha)$ is the Bessel function of

zero order.
We notice that in the static case, $\omega = 0$, the equations (1.2.7a) reduce to

$$\hat{p}_0(r) = \int_0^\infty \bar{\hat{p}}_0(s)\cdot s\cdot J_0(sr)ds = \begin{cases}\rho g\hat{w}_0 & (0 \le r < r_0)\\ 0 & (r > r_0)\end{cases} \tag{1.2.7b}$$

so that $\hat{p}_0(r) = \rho g\,\hat{w}_0$.
Further we notice that the asymptotic development of (1.2.7a) for large values of ω gives rise to the solution (Lamb (1945) p. 138)

$$\hat{p}_0(r) = -\tfrac{2}{\pi}\omega^2\rho\{r_0{}^2 - r^2\}^{1/2} \quad (0 \le r < r_0)$$

$$= 0 \qquad (r > 0) \tag{1.2.7c}$$

so that

$$\hat{P}_0 = -\tfrac{4}{3}\rho r_0{}^3\omega^2\hat{w}_0 \tag{1.2.7d}$$

without coefficients of damping and restoring force.

(Incompressible) homogeneous isotropic elastic half-space
Firstly, we consider the case in which a weightless rigid circular disk of radius r_0 is attached frictionless to the horizontal upper surface of a homogeneous, isotropic, elastic half-space and is asumed to undergo a vertical periodic displacement $w_0(t) = \hat{w}_0\, e^{i\omega t}$. The steady state solution of this boundary value problem is given by Awojobi and Grootenhuis (1965):

$$\begin{aligned}&\int_0^\infty \frac{\alpha_1}{\varphi(s)}\ \bar{\hat{\sigma}}_0(s)sJ_0(sr)ds = \frac{\mu\hat{w}_0}{\omega_2{}^2} \quad (0 < r < r_0)\\ &\hat{\sigma}_0(r) = \int_0^\infty \bar{\hat{\sigma}}_0(s)sJ_0(sr)ds = 0 \qquad (r > r_0)\end{aligned} \tag{1.2.8a}$$

with $\hat{\sigma}_0(r)$ the normal stress at the upper-surface with

$$\hat{\sigma}_0(r) = \int_0^\infty \bar{\hat{\sigma}}_0(s)sJ_0(sr)ds\,, \quad \bar{\hat{\sigma}}_0(s) = \int_0^\infty \hat{\sigma}_0(r)rJ_0(rs)dr \tag{1.2.8b}$$

and $\varphi(s)$ the Rayleigh function

$$\varphi(s) = (2s^2 - \omega_2{}^2)^2 - 4s^2(s^2 - \omega_1{}^2)^{1/2}(s^2 - \omega_2{}^2)^{1/2} \tag{1.2.8c}$$

with

$$\omega_1 = \frac{\omega}{c_1},\ \omega_2 = \frac{\omega}{c_2} \tag{1.2.8d}$$

$$\alpha_1{}^2 = s^2 - \omega_1{}^2 \tag{1.2.8e}$$

and

$$c_1{}^2 = \frac{2(1-\nu)\mu}{(1-2\nu)\rho},\quad c_2{}^2 = \frac{\mu}{\rho},\ \frac{c_1}{c_2} = \{\frac{2(1-\nu)}{(1-2\nu)}\}^{\frac{1}{2}} = \beta^* \tag{1.2.8f}$$

with c_1 and c_2 the velocity of the irrotational waves and equivoluminal waves, respectively. Further, μ is the shear modulus and ν is Poisson's ratio. c_1 is also the velocity of wave propagation in a laterally constrained bar. Awojobi (1971) considered the high frequency solution of (1.2.8). He showed that for the *compressible* homogeneous half-space

$$\begin{aligned} \hat{\sigma}_0(r) &\approx -i\beta^* \sqrt{\mu\cdot\rho}\cdot\hat{w}_0 \quad (0 < r < r_0) \\ &= 0 \quad (r > r_0) \end{aligned} \tag{1.2.9a}$$

with $\{\beta^* = \{2(1-\nu)/(1-2\nu)\}^{1/2}$, (1.2.9b)
so that *the theory of subgrade reaction is satisfied, this in contrast with low frequency factor and static stress contribution*. Thus we have

$$\left\{-\omega^2 M^\circ + i\beta^*\omega\pi r_0{}^2\sqrt{\mu\rho}\right\}\hat{w}_0 = \hat{P}_0 \tag{1.2.9c}$$

without added mass and restoring force. *This result corresponds to the case in which a laterally constrained semi-infinite rod suddenly has a velocity* $\dot{w}_0$ $(t = 0_+)$ *applied to the free end, because at time* $t =$ 0_+ *no wave propagation can take place.*
For the case of an *incompressible* subgrade Awojobi obtained

$$\begin{aligned} \hat{\sigma}_0(r) &\approx -\frac{2}{\pi}\rho\cdot\hat{w}_0\omega^2(r_0{}^2 - r^2)^{1/2} \quad (0 < r < r_0) \\ &= 0 \quad (r > r_0) \end{aligned} \tag{1.2.10a}$$

like in the case of deep water (compare (1.2.5b)). We notice that the static stress distribution is given by

$$\sigma_{0s}(r) = \frac{2\mu w_0}{\pi(1-\nu)}\{r_0{}^2 - r^2\}^{-1/2}).$$

Integration of (1.2.10a) over the loaded area $\pi r_0{}^2$ gives ultimately rise to

$$-\omega^2(M^\circ + M^*)\,\hat{w}_0 = \hat{P}_0 \quad, M^* = \frac{4}{3}\rho r_0{}^3 \tag{1.2.10b}$$

For the case of an infinitely long rigid strip base of width $2d_0$ we have for the *compressible* subgrade

$$(-\omega^2 M^\circ + i\beta^* \omega 2d_0 \sqrt{\mu\rho}\,\}\,\hat{w}_0 = \hat{P}_0 \tag{1.2.11}$$

and for the *incompressible* subgrade

$$-\omega^2(M^\circ + M^*)\hat{w}_0 = \hat{P}_0 \quad , M^* = \frac{\pi}{2}\rho d_0^2 \tag{1.2.12}$$

Bycroft (1977) slightly amended Awojobi's results. He writes for the cases of a weightless rigid circular base of radius r_0:

$$\hat{w}_0 = \frac{-\hat{P}_0}{\mu r_0}(f_1 + if_2),\ \hat{P}_0 = -\mu r_0 \frac{f_1 - if_2}{f_1^{\,2} + f_2^{\,2}} \cdot \hat{w}_0 \tag{1.2.13a}$$

He shows that for large values of

$$a_0 = \left\{\frac{\rho r_0^{\,3}\omega^2}{\mu r_0}\right\}^{1/2} \tag{1.2.13b}$$

in the case of a *compressible* subgrade

$$f_1 \approx -\frac{\kappa_v}{a_0^{\,2}} \quad , f_2 = \frac{1}{\pi a_0}\,\frac{1}{\beta^*} \tag{1.2.13c}$$

with κ_v varies from +0.09 to +0.06 as ν varies from ν = 0 to ν = 0.333, and in the case of an *incompressible* subgrade

$$f_1 \approx \frac{3}{4a_0^{\,2}} \quad , f_2 \approx \frac{1.2.93}{a_0^{\,3}} \tag{1.2.13d}$$

Thus, for the *compressible* subgrade we obtain (compare (1.2.9c))

$$\{-\omega^2 M^\circ + i\pi\beta^* r_0^{\,2}\omega\cdot\sqrt{\mu\rho}\}\,\hat{w}_0 + (\beta^*\pi)^2 \mu r_0 \kappa_v \hat{w}_0 = \hat{P}_0 \tag{1.2.14}$$

and for the incompressible subgrade (compare (1.2.10b))

$$[-\omega^2(M^\circ + \frac{4}{3}\rho r_0^{\,3}) + i\,3.43\,r_0^{\,2}\sqrt{\mu\rho}\ \ \omega]\hat{w}_0 = \hat{P}_0 \tag{1.2.15}$$

For the case of low frequencies we have the following results. In the case of an *incompressible* elastic subgrade the low frequency added mass of a rigid disk is about (3/2) times the high frequency added mass, and in the case of a *compressible* elastic half-space the low frequency added mass is about (1/2) times the high-frequency incompressible added mass (Bycroft (1956) and Awojobi and Grootenhuis (1965)). In Appendix 1.2 the reciprocity between the corresponding problems of a vertically loaded rigid strip (plane strain) and a rigid disk (axial symmetric) is presented.

The above results show that the high frequency added mass is only sigificant for incompressible subgrades (compare Wolf (1985)).

Finally, we consider the case in which a rigid sphere of radius r_0 is surrounded by a *compresible*, homogeneos, isotropic elastic medium of infinite extent with mass density ρ, shear modulus μ and Poisson's ratio ν.
We suppose that the rigid sphere is subjected to a vertical centric periodic force $P_0(t) = \hat{P}_0 e^{i\omega t}$, so that at the steady state the sphere undergoes a rigid displacement $w_0(t) = \hat{w}_0 e^{i\omega t}$. It is assumed that at the location of the surface of the sphere the displacement of the surrounding elastic space equals the displacement of the sphere. At this location the horizontal displacement is equal to zero and the vertical displacement is constant and equals $\hat{w}_0$. With

$$\alpha = i\omega\{(1-2\nu)\rho/2(1-\nu)\mu)\}^{1/2}\,,\ \ \beta = i\omega\{\rho/\mu\}^{1/2},\ \frac{\alpha}{\beta} = \{\frac{1-2\nu}{2(1-\nu)}\}^{\frac{1}{2}} = \frac{1}{\beta^*}$$

de Josselin de Jong (1956) finds that

$$2\mu\hat{w}_0 = \frac{\hat{P}}{4\pi(1-\nu)r_0} \cdot \frac{\{(1-2\nu)[3+3\beta r_0+\beta^2 r_0^2] + 4(1-\nu)[3+3\alpha r_0+\alpha^2 r_0^2]\}}{\{(1+\alpha r_0)[3+3\beta r_0+\beta^2 r_0^2] + 2(1+\beta r_0)[3+3\alpha r_0+\alpha^2 r_0^2]\}} \tag{1.2.16a}$$

$$\hat{P}_0 = 4\pi\mu r_0 \hat{w}_0 \left\{ \frac{1+\alpha r_0}{\frac{\alpha^2}{\beta^2} + \frac{2(3+3\alpha r_0+\alpha^2 r_0^2)}{3+3\beta r_0+\beta^2 r_0^2}} + \frac{2(1+\beta r_0)}{\frac{\alpha^2}{\beta^2}\cdot\frac{3+3\beta r_0+\beta^2 r_0^2}{3+3\alpha r_0+\alpha^2 r_0^2}+2} \right\} \tag{1.2.16b}$$

Now it is noteworthy (de Josselin de Jong (1956)) that the first term between the brackets at the right hand of (1.2.16b) represents the contribution I (say) of the irrotational wave (involving dilatation and distortion) and the second term the contribution R (say) of the equivoluminal (involving distortion and rotation) to the bearing power of the surrounding elastic space. In the case of high frequency $(I/R) = \beta^*/2$, and in the case of low frequency $(I/R) = 1/2$. In soils the low frequency (long) waves are destructive.
The classical solution to the problem of a vibrating rigid sphere surrounded by a *compressible* inviscid fluid, can be discovered from (1.2.16b) with $\mu = 0$, $\beta \to \infty$ but $\mu\beta^2 = -\omega^2\rho$ and $\beta^{*2}\mu = E_w$, the elasticity of fluid volume (Lamb (1945) p. 510)) (term I):

$$\hat{P}_0 = \frac{4}{3}\pi r_0^3\rho\omega^2 \left\{ -\frac{2+\kappa^2 r_0^2}{4+\kappa^4 r_0^4} + i\frac{\kappa^3 r_0^3}{4+\kappa^4 r_0^4} \right\} \hat{w}_0 \tag{1.2.16c}$$

with wave number $\kappa = \omega/c_w$, wave velocity $c_w = (E_w/\rho)^{1/2}$ of the irrotational waves with wave length $\lambda = 2\pi/\kappa$.
In the cases of long waves and incompressibility ($\kappa r_0 \to 0$) there is the added mass effect ($M^* = (2/3)\pi r_0^3$) and the damping vanishes. At high frequencies the damping dominates: $C = (4/3)\pi r_0^2 \rho c_w$.

Let us return to the formulae (1.2.16a) and (1.2.16b).
In the static case, $\omega = 0$,

$$P_0 = 24\pi\mu r_0 \frac{(1-\nu)}{(5-6\nu)} w_0 \tag{1.2.17a}$$

In the case of large frequency oscillations, i.e., $\omega \to \infty$, $\alpha \to \infty$ and $\beta \to \infty$ it follows that

$$\hat{P}_0 = 4\pi r_0^2 \left\{ \frac{1}{3} \left(\frac{2(1-\nu)}{1-2\nu}\right)^{1/2} (\mu\rho)^{\frac{1}{2}} + \frac{2}{3} (\mu\rho)^{1/2} \right\} i\omega \hat{w}_0 \tag{1.2.17b}$$

with no added mass (compare (1.2.16c)).
For the case of an *incompressible* medium we have $\nu = \frac{1}{2}$, so that $\alpha = 0$. Then the expression (1.2.16a) reduces to

$$2\mu \hat{w}_0 = \frac{\hat{P}_0}{2\pi r_0} \frac{6}{\{9 + 9\beta r_0 + \beta^2 r_0^2\}} \tag{1.2.18a}$$

or

$$\hat{P}_0 = \frac{2\pi\mu r_0}{3} \{9 + 9\beta r_0 + \beta^2 r_0^2\} \hat{w}_0 , \ \beta = i\omega\{\rho/\mu\}^{1/2} \tag{1.2.18b}$$

In the case of high frequency oscillations it follows that

$$\hat{P}_0 = -\frac{2\pi\rho r_0^3}{3} \omega^2 \hat{w}_0 , \ M^* = \frac{2\pi\rho r_0^3}{3} \tag{1.2.19}$$

This value of the added mass corresponds to the value of the added mass of a sphere moving through an incompressible perfect fluid of infinite extent. In the static case, $\omega = 0$, we have

$$P_0 = 6\pi\mu r_0 w_0 \tag{1.2.20}$$

The static stress distribution corresponds with the stress distribution belonging to the slow steady flow of an incompressible viscid fluid past a sphere under no-slipping conditions. With w the uniform fluid velocity at infinity and μ the viscosity, the drag force on the sphere is given by $P = 6\pi\mu r_0 w$ into the direction of the flow (Lamb (1945), p. 597).
We notice that at the location of a horizontal plane through the center of the rigid sphere the stress component in the horizontal radial direction, the stress component in the horizontal circumferential direction and the stress component in the vertical direction are equal to zero. Unfortunately, on that plane the shear stress is not equal to zero. At that plane simple shear occurs (compare the slow motion of a sphere through incompresible viscid fluid). However, at large values of the frequency ω this shear stress may be disregarded since it behaves as $(1/\omega)$ when $\omega \to \infty$, and the stress at the surface of the sphere becomes isotropic.
As a consequence of the zero normal stresses at the location of the horizontal plane through the center of the sphere, and the vanishing of the shear stress on that plane for large values of the frequency, we may obtain the following result.
When a rigid hemisphere of radius r_0 is embedded in an *incompressible* homogeneous elastic half-space such that the upper-surface is flat (Figure 1.1) and is loaded by a vertical impulsive force $\widetilde{P}(t$

= 0) the hemisphere obtains the initial velocity

$$\tilde{P}(t = 0_+) = (M^* + M^\circ) w_0(t = 0_+), \; M^\circ = (\pi/3)\rho r_0^3 \tag{1.2.21}$$

For the case of an infinitely long semi-cylinder the added mass M° is given by $(\pi/2)\rho d_0^2$ per unit length of the cylinder.

When we divide the left-hand of the expression (1.2.16a) by 2, we obtain the expression of the high-frequency motion of an embedded hemisphere, a result that holds even as to a first approximation for low frequencies (compare F. Medina in Gazetas (1985)).

Gibson soil

As a consequence of the mathematical analogy between the equations of the irrotational deep water motions and the equations of the irrotational Gibson soil motion, we have the following results.

The high frequency isotropic stress distribution under a rigid disk (a rigid strip) at the upper surface of the Gibson soil has exactly the same form as the pressure distribution under a rigid disk (a rigid strip) at the upper surface of deep water (and an incompressible homogeneous elastic half-space). Therefore, the high frequency oscillation of a rigid disk with radius r_0 (a rigid strip) generates a high frequency added mass $M^* = (4/3)\rho r_0^3$ ($M^* = (\pi/2)\rho r_0^2$).

We notice that the exact solution to the rigid disk problem is given by Awojobi (1973, 1974) and Kruijtzer (1976):

$$\int_0^\infty \frac{\bar{\hat{\sigma}}_0(s) J_0(sr)}{(\rho g + 2m) - \dfrac{\rho\omega^2}{s}} ds = -\hat{w}_0 \quad (0 \le r < r_0)$$

$$\hat{\sigma}_0(r) = \int_0^\infty \bar{\hat{\sigma}}_0(s)\, s J_0(sr) ds = 0 \quad (r \ge r_0)$$

with $\bar{\hat{\sigma}}_0(s)$ the Hankel transform of $\hat{\sigma}_0(r)$ (compare (1.2.7a)). (Awojobi did not take the gravity into account). The upper limit of the low frequency added mass is given in Appendix 1.3.

In a recent paper (Kruijtzer (2001)) we have considered the case in which a semi-spherical (or -cylindrical) rigid punch is pressed against the upper surface of a Gibson soil in such a way that the resulting upper surface including the punch becomes flat. At the location of the deformed upper surface the shear modulus is equal to zero, but the stress at this location is not isotropic. In the linearized theory the principle of stress superposition is valid.

When we require that at the location of the surface of the hemisphere (semi-cylinder) the displacement of the Gibson soil is equal to $w_0(t)$ we have for large frequency factors the added mass $(1/3)\pi\rho r_0^3$ for the hemisphere and the added mass $(\pi/2)\rho d_0^2$ per unit length for the semi-cylinder with an isotropic stress at the surface of the hemisphere or semi-cylinder.

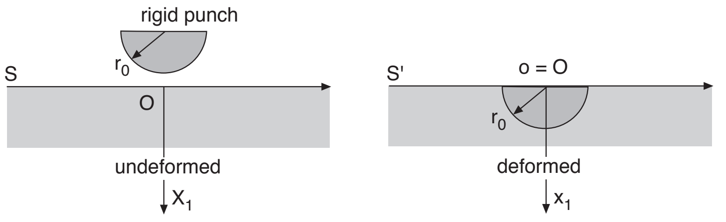

Figure 1.4. Pressed rigid punch.

Water saturated porous elastic space

It is assumed that the porous elastic space is homogeneous and isotropic. Further, it is assumed that the solid material behaves as being perfectly rigid under action of the all round water pressure. In the *fully undrained* state the solid and the water possess equal velocities. The velocity c_1 of the irrotational waves and the velocity c_2 the equivoluminal waves are given by

$$c_1 = \left\{ \frac{\beta^{*2}\mu + E_w / n}{(1-n)\rho_s + n\rho_w} \right\}^{1/2}, \quad c_2 = \left\{ \frac{\mu}{(1-n)\rho_s + n\rho_w} \right\}^{1/2}$$

with n the solid porosity, ρ_s the solid mass density, ρ_w the water mass density. Further, E_w is the elasticity of water volume, μ is the solid shear modulus and $\beta^{*2}\mu$ the elasticity of a laterally constrained solid volume.

When it is assumed that the water is *incompressible, the undrained composite* behaves as being an *incompressible elastic space*. High frequency vibrations generate added mass showing that high frequency pile driving is useless (de Josselin de Jong (1956)).

In the *fully drained* state there is no excess of water pressure and the elastic composition behaves as being an elastic space. It should be noticed that the fully drained and undrained states must be satisfied on the boundary.

1.3. Concluding remarks

The linear analysis of some footing and floating body problems shows the correspondence between the high frequency added in-phase subgrade masses for the cases of deep water, a Gibson soil and an incompressible homogeneous elastic half-space. High frequency states may occur when the rigidity of the subgrade is small, the dimensions of the loaded surface area are large, the weight of the foundation is large or the frequency input is high such as occurs on impact, blast loading, vibrating machines, the deceleration of bodies hitting the ground, high frequency components of earthquakes and running on a thin ice layer. With an increase of frequency of a surface loading the depth of penetration of resulting effect in the subgrade decreases. Furthermore, in the case of large frequency factors, an increase of compressibility of the subgrade decreases the effect of added mass and increases the effect of damping. Furthermore, in the case of high-frequencies the compressible subgrade reacts like a uniform bed of dashpots.

In the presentation of the solutions to different cases involving the motion of a rigid mass $M°$ connected with a massless Kelvin-Voigt element with spring constant K and the damping constant C, it is conventional to use the dimensionless parameters C/C_c and ω/ω_n. Here ω_n is the eigenfrequency in the absence of damping, $\omega_n = (K/M°)^{1/2}$, and C_c is the critical damping, $C_c = 2(KM°)^{1/2}$. The parameters separate the effects of damping and frequency, but it is cumbersome to determine the effects of changes in $M°$ and K. Lysmer (Richart, Woods and Hall (1970), p. 31) separated the effects of mass and frequency by introducing the dimensionless parameters

$$\bar{a}_0{}^2 = \frac{\omega^2 \cdot C^2}{K^2}, \; \bar{b} = \frac{M°K}{C^2}; \; \bar{b}\bar{a}_0{}^2 = \frac{M_0\omega^2}{K} = \frac{\omega^2}{\omega_n{}^2} \tag{1.3.1a}$$

In the theory of vertical oscillation of footings with circular base resting on the surface of an (incompressible) homogeneous elastic half-space the dimensionless parameters

$$a_0{}^2 = \frac{\omega^2 \cdot \rho r_0{}^3}{\mu r_0}, \; b_0 = \frac{M°}{\rho r_0{}^3} \tag{1.3.1b}$$

appear in a natural way, with a_0 is the dimensionless frequency and b_0 is the mass ratio.
In the case of a Gibson soil we introduce the dimensionless quantities

$$u_0{}^2 = \frac{\rho r_0{}^3 \omega^2}{(\rho g + 2m) r_0{}^2}, \; b_0 = \frac{M°}{\rho r_0{}^3} \tag{1.3.1c}$$

with r_0 the radius of the loaded area.
In the rather extensive litterature the differential equation of the vertical motion of a rigid footing with a flat circular base of radius r_0 attached to the horizontal upper surface of a compressible homogeneous isotropic elastic half-space is presented in the form (Richart et al (1970), p. 208):

$$(M° + M^*)\ddot{w}(t) + \frac{3.46\, r_0{}^2}{1-\nu}\sqrt{\rho\mu}\,\dot{w}(t) + \frac{4\mu r_0}{1-\nu} w(t) = P_0(t), \quad 0 \le a_0 < 1 \tag{1.3.2a}$$

so that for the case of an incompressible half-space

$$(M° + M^*)\ddot{w}_0(t) + 6.92 r_0^2 \sqrt{\rho\mu}\,\dot{w}(t) + 8\mu r_0 w_0(t) = P_0(t) \quad 0 \le a_0 < 1 \tag{1.3.2b}$$

With $P_0(t) = \hat{P}_0\, e^{i\omega t}$ and $w_0(t) = \hat{w}_0 e^{i\omega t}$ the equation (1.3.2b) reduces to the steady state equation of motion

$$-\omega^2(M^\circ + M^*)\hat{w}_0 + i\omega\; 6.92\, r_0{}^2\sqrt{\mu\rho}\,\hat{w}_0 + 8\mu r_0 \hat{w}_0 = \hat{P}_0, \; 0 \le a_0 < 1 \tag{1.3.2c}$$

Our collected and derived results show that these lumped equations are not valid in the case of high frequency factors. In the case of a compressible subgrade the damping term is dominant and in the case of an incompressible subgrade the mass term is dominating at large forced frequencies.
We notice that for values of $(((1-\nu)b_0/4) > 2$ the equation $(((1-\nu)b_0/4)a_0{}^2 = 1$ determines with sufficient accuracy the value of the resonance frequency ω_n. Further, we notice that the equalization $\bar{b} = b_0(1-\nu)/4$ and $\bar{a}_0 = a_0$ (compare (1.1a) and (1.1b) gives rise to an indication of the value of the damping coefficient.

In the subjoined table the results of this treatise have been summarized. It should be noticed that the displacement $\hat{w}_0 e^{i\omega t}$ due to the additional loading$\hat{P}_0 e^{i\omega t}$ must be added to the initial static diplacement due to the own weight (and other initially static loadings) of the footing or the floating body. Further, it should be noticed that the given low frequency added mass represents only an indicative estimation. The corresponding plane strain (2D) results may be found in the text.

Table. Summary of the results.

<table>
<tr><td colspan="3">Vertical motion of a footing or floating body with mass M°:
$(M^\circ + \bar{M}^*)\, \ddot{w}_0(t) + \bar{C}\, \dot{w}_0(t) + \bar{K}\, w_0(t) = P_0(t)$
with $P_0(t) = \hat{P}_0 e^{i\omega t}$, $w(t) = \hat{w}_0 e^{i\omega t}$ $-\omega^2(M^\circ + M^*(\omega))\hat{w}_0 + i\,\omega\, C(\omega)\,\hat{w}_0 + K(\omega)\,\hat{w}_0 = \hat{P}_0$
$\bar{M}^*$, $M^*(\omega)$ = added mass; $\bar{C}$, $C(\omega)$ = coefficient of geometrical damping (wave radiation); $\bar{K}$, $K(\omega)$ = coefficient of restoring force</td></tr>
<tr><td colspan="3">Reaction of a homogeneous, isotropic, elastic half-space;
shear modulus μ, Poisson's ratio ν, mass density ρ.</td></tr>
<tr><td></td><td>incompressible</td><td>compressible $\beta^* = \left\{\frac{2(1-\nu)}{(1-2\nu)}\right\}^{1/2}$
$\kappa_\nu < 0.1$</td></tr>
<tr><td>weightless disk
high frequency</td><td>$M^* = (4/3)\,\rho r_0^3$
$C = 3{,}43\, r_0^2 \sqrt{\mu\rho}$
$K \approx 0$</td><td>$M^* = 0$
$C = \pi\beta^* r_0^2 \sqrt{\mu\rho}$
$K = \kappa_\nu(\pi\beta^*)^2\, \mu r_0$</td></tr>
<tr><td>weightless disk
low frequency</td><td>$M^* = (3/2)(4/3)\,\rho r_0^3$
$C = 6.92\, r_0^2 \sqrt{\mu\rho}$
$K = 8\mu r_0$ (static)</td><td>$M^* = (1/2)(4/3)\,\rho r_0^3$ $(\nu = 0)$
$C = 3.46\, r_0^2 \sqrt{\mu\rho}$ $(\nu = 0)$
$K = 4\mu r_0$ (static) $(\nu = 0)$</td></tr>
<tr><td>weightless hemisphere
high frequency</td><td>$M^* = (\pi/3)\,\rho r_0^3$
$C = 3\pi\, r_0^2 \sqrt{\mu\rho}$
$K \approx 0$</td><td>$M^* \approx 0$
$C = (2\pi/3) r_0^2 (\beta^* + 2)\sqrt{\mu\rho}$
$K \approx 0$</td></tr>
<tr><td colspan="3">Reaction of deep water; mass density ρ, acceleration due to gravity g.</td></tr>
<tr><td>weightless disk (shallow pontoon)
high frequency</td><td colspan="2">$M^* = (4/3)\,\rho r_0^3$, $M^\circ = \rho\pi r_0^2 \cdot h$, h draft of pontoon at rest
$C \approx 0$
$K \approx 0$</td></tr>
<tr><td>weightless disk (shallow pontoon)
low frequency</td><td colspan="2">$M^* \approx (3/2)(4/3)\,\rho r_0^3$, $M^\circ = \rho\pi r_0^2 \cdot h$
$C \approx 0$
$K \approx \pi r_0^2\, \rho g$</td></tr>
<tr><td>hemisphere
high frequency</td><td colspan="2">$M^* = (\pi/3)\,\rho r_0^3$, $M^\circ = (2\pi/3)\rho r_0^3$
$C \approx 0$
$K \approx 0$</td></tr>
<tr><td>hemisphere
low frequency</td><td colspan="2">$M^* = (3/2)(\pi/3)\,\rho r_0^3$, $M^\circ = (2\pi/3)\rho r_0^3$
$C \approx 0$
$K \approx \pi r_0^2 \rho g$</td></tr>
<tr><td colspan="3">Reaction of a Gibson soil; shear modulus m times depth, mass density ρ.</td></tr>
</table>

weightless disk high frequency	$M^* = (4/3)\,\rho r_0^3$ $C \approx 0$ $K \approx 0$
weightless disk low frequency	$M^* \approx (3/2)(4/3)\,\rho r_0^3$ $C \approx 0$ $K \approx \pi r_0^2\,(\rho g + 2m)$
pressed hemisphere high frequency	$M^* = (\pi/3)\,\rho r_0^3$ $C \approx 0$ $K \approx 0$
Added mass, coeffcient of geometrical damping and coeffcient of restoring force for the additional vertical motion of footings and floating bodies, which motion should be added to the static initial state.	

Our results hopefully bridge the gap between the mechanics of deep water and of an elastic substratum, and the gap between low frequency and high frequency problems with respect to the compressibility of the substrata.

Appendix 1.1

G.N. Bycroft (1956) showed that in the case the upper surface of a homogeneous incompressible isotropic elastic half-space is subjected by an almost uniform displacement over a circular loaded area with radius r_o, the relationship between the exciting force $P(t) = \hat{P}e^{i\omega t}$ and the vertical displacement $w(t)$ is given by

$$w = w(t) = -\frac{\hat{P}e^{i\omega t}}{\mu r_0}(f_1 + if_2) \qquad \text{(a.1.1)}$$

where f_1 and f_2 are two functions of frequency being effectively the in-phase and out-phase components of the displacement w of the weightless disk of radius r_0 (Figure a.1). In the case of very high frequencies ω (Awojobi (1971), Bycroft (1977)):

$$f_1 \approx \frac{\mu r_0}{\frac{4}{3}\rho r_0^3 \omega^2},\; f_2 \approx \frac{1.93}{r_0^3 \cdot \omega^3 \cdot (\rho/\mu)^{3/2}} \qquad \text{(a.1.2)}$$

or with

$$a_0 = r_0(\omega\sqrt{\rho/\mu}) \qquad \text{(a.1.3)}$$

$$f_1 \approx \frac{3}{4a_0^2},\; f_2 \approx \frac{1.93}{a_0^3} \qquad \text{(a.1.4)}$$

It is noticed that the asymptotic expression of f_2 has been given by Bycroft (1977).
Further, in the static case, i.e. $\omega = 0$, we have