9 3D Plots of Functions

Plots in three dimensions (3D plots) in Mathcad are similar to 2D plots. However, there are differences of which you should be aware. Mathcad has two kinds of 3D plot:
(1) the contour plot (a 2D 'altitude' map)
(2) the true 3D plot with three axes.

Contour Plot

The contour plot is an altitude map. It plots lines of equal height on a flat plane. You will find it under Plots > Insert Plot. We illustrate its use with a paraboloid:

$$z(x,y):=x^2+y^2$$

This has its lowest point (zero) in the origin - it rises with an increasing slope as the distance from the origin increases. The colour denotes the local height z of the function - the z-scale is that at the bottom. The default ranges are

$$-10<x<10 \qquad -10<y<10 \qquad z \quad \text{auto}$$

You can change all three ranges plotted by editing the first and last tick values, and the tick distance by the choice of the second tick value. You can also choose between several colouring schemes and add values to the contour lines. Copy the plot to the right and try.
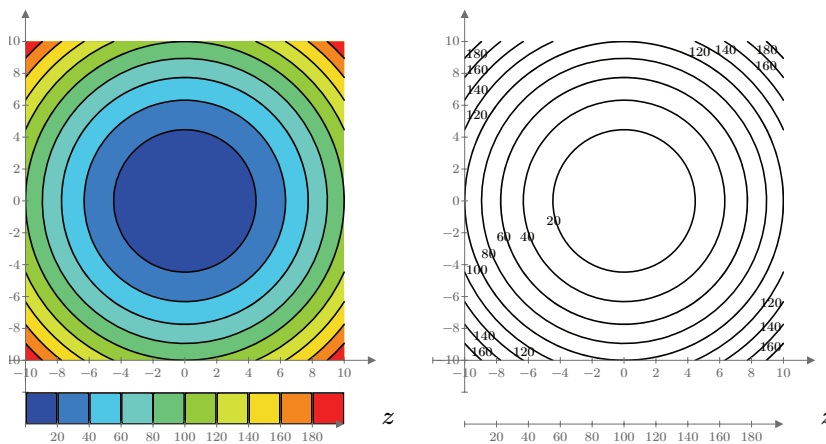
The contour plot has a little quirk. In the placeholder in the lower right corner, you must enter the name of the function to be plotted (here z), but *not* the function variables (here x and y). In other Mathcad plots you can also enter the function with variables.

True 3D Plots

Before looking at examples of true 3D plots, I first introduce a few terms. Below is an *image* of a true 3D plot that has been activated by clicking in it.
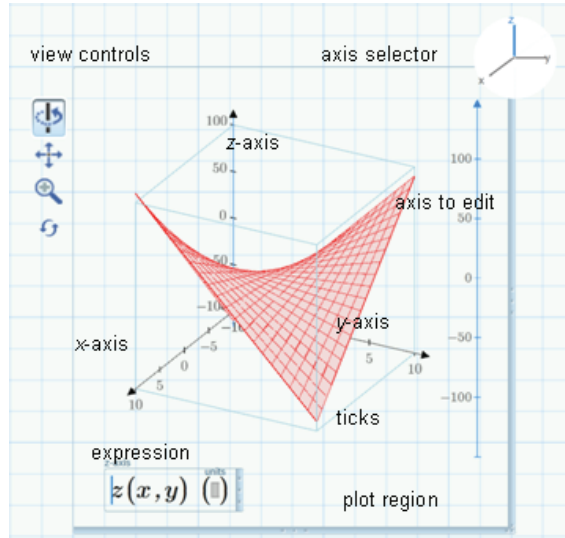
The plot has three axes: the x-, y- and z-axis. The z-axis is often vertical. The trace (here a red 'wireframe' with a pink 'surface') is in the rectangular volume bounded by the axes. In 2D plots, there are separate legend boxes for the x- and y-expressions. This is not so here: there is only a box for the z-axis. (There can be several of these: you can have two or more functions in the same plot.)

There is an axis selector in the upper right hand corner. An axis selected will turn blue, both in the selector and in the plot. You can then change the first, second and last tick values, just as in 2D plots. In this way, you can change the range plotted and the number of ticks.

You can move, squeeze and expand the plot with handles on the borders of the plot region. You can move, spin and tilt the plot with the view controls in the upper left corner, as we shall see in a moment.

A Paraboloid

We are going to make a plot of our paraboloid. Click in the upper left of the blank region and insert a plot using Plots > Traces > Insert Plot > 3D Plot.

Type **z(x,y** in the placeholder, click in a blank region, and a default plot appears. Activate the plot by clicking in it. The first view control that you should look for is 'Reset View' - something like 'Undo'. You may need it further on.

(In the book, I have left this plot in place.)
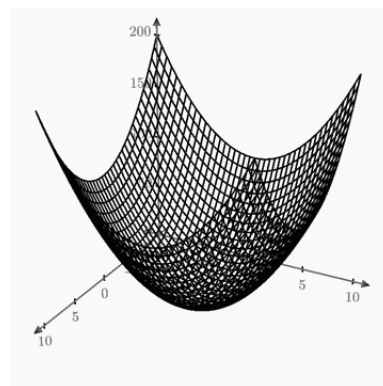
$$z(x,y)$$

Select 'Spin' (if it isn't already se-
lected) and drag in the plot. You can
turn and tilt the axes to view the func-
tion from all sides. When you have
finished, press the reset. Try the 'Pan'
and 'Zoom' buttons - they are fairly
obvious. Then reset again.

Click on the z-axis in the axis selector.
Change the value of the last (top) tick
value from 200 to 400, then click in a
blank region to see the effect. You will
have to change this back to 200 your-
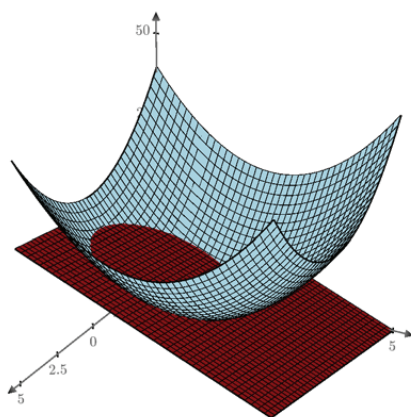self: you cannot do that with the reset
button.

In a second plot, I have changed the
trace colour and added a surface fill.
Try these yourself.
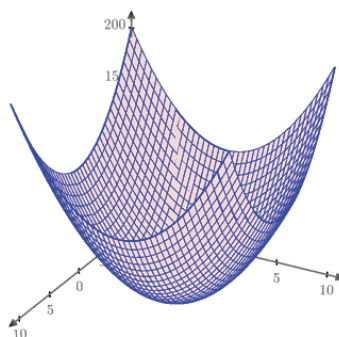
z

Two Functions $z_1(x,y) := x^2 + y^2 - 10$ $z_2(x,y) := x - 3 \cdot y - 5$

To plot a second function, select the legend and add a trace using Plots >
Traces > Add Trace. Below I have plotted a paraboloid and a sloping plane:
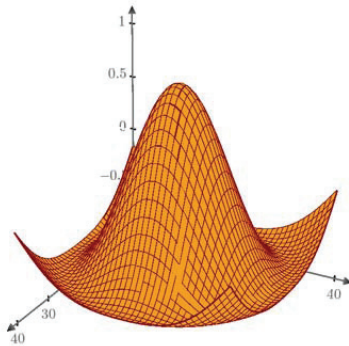
$z_1(x,y)$

$z_2(x,y)$

The two functions have contrasting colours, so you can see their inter-
section. Spin the plot to see what the shape is.

Your Own Matrix

We made our 2D plots using vectors. Something similar can be done with 3D plots, but here we need a matrix of z-values. I will show how this works using a function known as the 'Mexican hat'.

$$i := 0 .. 40 \qquad x_i := -1 + \frac{i}{20} \qquad j := 0 .. 40 \qquad y_j := -1 + \frac{j}{20}$$

$$Z_{i,j} := \frac{\cos\left(\pi \cdot \sqrt{x_i^2 + y_j^2}\right)}{\sqrt{x_i^2 + y_j^2 + 1}}$$

$$Z_{i,j}$$

Note that the horizontal tick values are those of i and j, not those of x and y.

A Sphere

Making a 3D plot using parameters is more difficult than in 2D because you can only enter z-values in the plot. I will illustrate how to do this by plotting a sphere. The calculations behind this are complicated, but Mathcad provides a function CreateMesh that does most of the work for you. The parametric equations of a sphere are:

$$F(\phi, \theta) := \begin{bmatrix} \sin(\phi) \cdot \cos(\theta) \\ \sin(\phi) \cdot \sin(\theta) \\ \cos(\phi) \end{bmatrix}$$

The parameter ϕ is a vertical angle of rotation; the parameter θ a horizontal angle of rotation. The required ranges and numbers of points are:

$$\phi_{min} := 0 \qquad \phi_{max} := \pi \qquad n_\phi := 18$$
$$\theta_{min} := 0 \qquad \theta_{max} := 2 \cdot \pi \qquad n_\theta := 36$$

The set of matrices is constructed using the CreateMesh function:

$$M := \text{CreateMesh}\left(F, \phi_{min}, \phi_{max}, \theta_{min}, \theta_{max}, n_\phi, n_\theta\right)$$

Insert this in the placeholder of a 3D plot and you get a sphere. You can rotate and tilt the sphere like any other function. If you want to use this technique for other functions, you can best copy and modify my calculations.
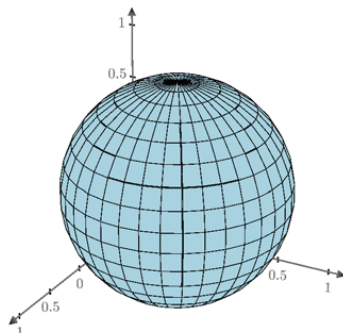
M

Summary
(1) Mathcad has two kinds of 3D plots: contour plots and true 3D plots. They are entered via Plots > Insert Plot...
(2) Contour plots are like a map with altitude lines.
(3) True 3D plots are similar to 2D plots, but they have three axes. Axes are selected with a selector and edited separately. The range and tick distance change by editing the first, second and last tick values.
(4) Select a plot with [Ctrl][Click]. Move, squeeze or expand the plot with the handles on the borders of the plot region.
(5) To quickly make a 3D plot, define the function $z(x,y)$, make a plot and type the function in the placeholder.
(6) You can also form a matrix of z-values and type the matrix name in the placeholder.
(7) If you want to try 3D parametric plots, copy my example and modify the equations.

9-1 Spatial Symmetry   5 min
You are to construct four 3D plots. Insert a 3D plot below the equation and enter the name of the equation in the placeholder. The turn and tilt the surface to see what it looks like.

Plane $s(x,y):=-x-y$

Paraboloid $p(x,y):=x^2+y^2$

Hyperboloid $h(x,y):=x^2-y^2$

Wave $w(x,y):=\sin\left(\dfrac{x}{\pi}\right)\cdot\cos\left(\dfrac{y}{\pi}\right)$

9-2 Leaf of Descartes   10 min

You are to study an interesting function that is known as the leaf of Descartes:

$$z(x,y):=x^3+y^3-3\cdot x\cdot y$$

The function looks like a steep slope, rising from the lower left upwards to the upper right. There is a little valley running into the slope, with a shallow pond in the valley.

Insert a contour plot below and type **z** (no parameters) in the placeholder. Then edit the ranges of both the *x* and *y* axis to get -2,-1..3. Click in the plot and enable Plot > Styles > Contour Values. Numbers appear on the countour lines. The line with a contour value of zero has the shape of a leaf, hence the name of the function.

Now make a 3D plot of the same function, and edit the *x* and *y*-axes to get the same scales as above. Turn and tilt the surface to see whether you recognise it, and whether you can see the 'pond'.

9-3 A Donut   10 min

You are to plot a donut (torus in math terms). Copy my plotting of a sphere and change the parametric equations to:

$$F(\phi,\theta):=\begin{bmatrix}(2+\cos(\phi))\cdot\cos(\theta)\\(2+\cos(\phi))\cdot\sin(\theta)\\\sin(\phi)\end{bmatrix}$$

You will get only get the upper half of a donut, and one that is too high. Double the range of ϕ and increase the range of the vertical axis. Then tilt your donut to have a good look at it.

9-4 Plot of a Spring   2 min

In this exercise you are to plot a line in space. Such a line is described by a single parameter. This in contrast to surfaces like the sphere that you have seen earlier. The coordinates of a spring can be described by:

$$S(t):=\begin{bmatrix}\sin(t)\\\cos(t)\\t\end{bmatrix}$$

Insert a 3D plot and type **S(t** in the placeholder. Then modify the settings until you like your 'spring'.

10 Units in Mathcad

One of the best ways of checking calculations is to keep track of the units. Mathcad can do much of this for you. When you begin this lesson, check in Math > Units that the Unit System is SI.

Variables with Units

You can give any constant a unit by multiplying the value and entering the unit in the placeholder:

width $\qquad W := 0.23 \cdot \big|$ $\qquad$ type as **W:0.23*[Enter]**

$\qquad\qquad W := 0.23 \cdot m$ $\qquad$ obtain from Math > Units > Length or type **m**

Math > Units >... contains perhaps all units that you will ever need. You can simply type the more common units, such as **m** for metre, **kg** for kilogram, **s** for second.... (The gram is not **g**, but **gm**. to avoid a clash with the gravitational constant **g**.) Our next variable is:

length $\qquad L := 0.45 \cdot m$

You can use these in calculations and automatically get the result with units:

area $\qquad A := W \cdot L \qquad A = 0.104 \ m^2$

Working with units is usually straightforward and quite natural. However, you have to be careful when using the same symbol for a unit and for a variable. I will illustrate this with electrical power:

electrical potential in Volt $\qquad E := 3.2 \cdot V$
electrical current in Ampere $\qquad I := 625 \cdot A \qquad I = 64.688 \ m^2$

This is wrong: Mathcad is using the area A that we calculated earlier instead of the unit Ampere. Note that A is not blue. To remove the problem change the *label* of the unit. Select the A in the unit placeholder and press Math > Style > Labels > Unit. You could also have avoided the problem by entering the A from Math > Units > Current. Now:

$\qquad\qquad\qquad I := 625 \cdot A \qquad I = 625 \ A$
The electrical power is $\qquad P := E \cdot I \qquad P = (2 \cdot 10^3) \ W$

Here we get the correct result, with a blue W standing for Watt. Also the width still has the correct unit:

$$W = 0.23 \ m$$

Compound units are also allowed: $\qquad v := 7.2 \cdot \dfrac{m}{s} \qquad v = 7.2 \ \dfrac{m}{s}$

Velocity times force should give power again: $F := 3 \cdot N$
$$v \cdot F = 21.6 \ W$$

You can change the output to any of the units that Mathcad knows:
$A = 0.104 \ m^2$ select the unit, delete and replace by **cm^2**
$A = \left(1.035 \cdot 10^3\right) \ cm^2$
Note that you have to *delete* the old unit before typing in the new one.

The unit specified can be a combination. Say you want the following velocity in kilometres (km) per hour (hr):

$$v := 300 \cdot \frac{m}{s} \qquad\qquad v = \left(1.08 \cdot 10^3\right) \frac{km}{hr}$$

One last example. Suppose an enthalpy of vaporisation is given as

$$\Delta H_V := 40 \cdot 10^3 \cdot kg \cdot m^2 \cdot s^{-2} \cdot mol^{-1}$$

You think the unit should be J/mol. In the evaluation below I have re-placed the whole group of units with **J/mol**:

$$\Delta H_V = \left(40 \cdot 10^3\right) \ J \cdot mol^{-1}$$

This shows that the two ways of writing are identical (although it does not look like that.)

Labels
You have just used a label to help distinguish between two uses of the letter *A*: as the *unit* Ampere and as the *variable* area. There are two ribbon entries dealing with labels. The first one (which you have just used) is under Math > Style... It changes a label locally (so the label of single variable, function or constant). The second is found under Formatting > Label Styles: it changes the looks of a certain label throughout a whole document. Don't get these two mixed up.

First open the Math > Style > Labels entry. You will see a list of the labels used by Mathcad. Close the list, but keep the Math ribbon open. Click in the different parts of the equations below and see the effect on the labels sub-entry.
$W := 0.23 \cdot m \qquad P := E \cdot I \qquad P = \left(2 \cdot 10^3\right) \ W$
The first *W* turns out to be a Variable. The 0.23 has no label; the *m* has the label Unit. *P* is again labelled as a variable, but the variables behind the assign operation are not labelled. The *W* in the last equation is again a unit. Because they have different labels, the two *W*'s behave as different variables.

Now open Formatting > Label Styles. Click on the last W in the equations, then open the list in Label Styles. Click on Unit, remove 'italic' by pressing the I-button and change the colour to red. You will see all units change throughout this document. Change them back to the default with the button with the turning arrows.

We have used the W above for two purposes: as unit and as variable. When you type this letter, Mathcad assumes that you want to use the last definition. Suppose you want to enter a new variable with the unit Watt. You get:

$$P_2 := 400 \cdot W \qquad P_2 = 92 \ m \qquad \text{wrong!}$$

To correct this, you can do two things. First, you can change the label of the W:

$$P_2 := 400 \cdot W \qquad P_2 = 400 \ W$$

This only works locally. If you want to get the unit all the time when typing you can define again:

$$W := 1 \cdot W \qquad P_3 := 500 \cdot W$$

Constants with Units

Mathcad contains a collection of the most important constants from physics under Math > Constants. In the following examples I have brought some units in the form as they are usually written:

Avogadro's constant $\qquad\qquad N_A = \left(6.022 \cdot 10^{23}\right) \ mol^{-1}$

Boltzmann constant $\qquad\qquad k = \left(1.381 \cdot 10^{-23}\right) \ J \cdot K^{-1}$

charge of the electron $\qquad\qquad e_c = \left(1.602 \cdot 10^{-19}\right) \ C$

gas constant $\qquad\qquad R = 8.314 \ J \cdot mol^{-1} \cdot K^{-1}$

Planck constant $\qquad\qquad h = \left(6.626 \cdot 10^{-34}\right) \ J \cdot s$

One constant which is not in the Mathcad list is:

gravitational acceleration $\qquad g = 9.807 \ m \cdot s^{-2}$

Avoiding Units

In some calculations (especially those involving solve blocks) units can slow things down and can give complications. Then you may wish to avoid units. Simply don't specify them!

If you do use units, do that throughout the whole worksheet. Having some variables with and some without units is asking for trouble. This may require redefining the occasional built-in constant. For example, if you use the velocity of light in vacuum, and want to avoid units:

$$c = \left(2.998 \cdot 10^8\right) \frac{m}{s} \qquad \text{from Mathcad}$$

becomes $c := 2.998 \cdot 10^8$ $c = 2.998 \cdot 10^8$

(You have to copy the number and paste it into the new expression.)

Summary
(1) To give a constant a unit, type * behind it, then enter the unit in the placeholder. You can either type the unit, or enter it via Math > Units.
(2) If you do not enter units in an assignment, Mathcad assumes that the variable in question has no unit.
(3) When you type the unit, you may be using the same symbol as a variable. The 'unit' then does not take on the blue format when you click outside the expression. If this happens, you have to change the label.
(4) To change a label, click in the offending constant, variable, unit or whatever, then go to Math > Style > Labels to choose the correct label.
(5) To reset the unit as the default while typing, enter:

$m := m \qquad s := s \qquad N := N \qquad$ or whatever unit you are using

(6) Units can often be expressed in several ways.

$F := 1 \cdot N \qquad\qquad F = 1 \ N \qquad F = 1 \ kg \cdot m \cdot s^{-2}$

(7) If you do not like the way that Mathcad shows a unit, type *your* form in the unit placeholder. If your form is wrong, Mathcad will correct it.
(8) Either use units consequently throughout a worksheet, or do not use them at all.

10-0 Using Units - Starting   5 min

This is a series of exercises using units. They range from simple to difficult:

    (1) Food in the Body
    (2) Dissolution of a Tablet
    (3) Plate Viscometer
    (4) Temperature in a Spray Dryer
    (5) Volume of an Ideal Gas
    (6) Falling Particle
    (7) Flow From a Vessel
    (8) Concentrations in a Mixture.
    (9) Vapour Pressure of Water

Before we begin the exercise, we first set up our system. Check in Math >
Units that the Units System is SI. Then go to Formatting > Label Styles
and click in the unit **m** below

$$D := 0.1 \cdot \mathbf{m}$$

Make the symbol for units to be upright (not italic), bold and red. This will
allow you to clearly distinguish between variables and units. Perhaps this is
a good point to tell that the standard notation for units is in ordinary black
upright letters - so not the format used now or by Mathcad.

10-1 Food in the Body   5 min

Carbohydrates are burned in the body. The heat flow released is
given by:

$$Q = r \cdot \Delta H_C$$

    rate of burning $r := 0.4 \cdot \mathbf{kg} \cdot \mathbf{day}^{-1}$
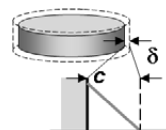   enthalpy of combustion $\Delta H_C := 20 \cdot 10^6 \cdot \mathbf{J} \cdot \mathbf{kg}^{-1}$

Calculate the heat flow and check that the unit is what you expect.

10-2 Dissolution of a Tablet   5 min

The mass flux out of a dissolving tablet is given by:

$$j = c \cdot \frac{Đ}{\delta}$$

The parameters have values:

    solubility of the drug $c := 0.4 \cdot \mathbf{kg} \cdot \mathbf{m}^{-3}$
    diffusivity of the drug in water $Đ := 0.6 \cdot 10^{-9} \cdot \mathbf{m}^2 \cdot \mathbf{s}^{-1}$
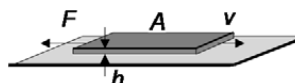   thickness of the diffusion film $\delta := 10^{-4} \cdot \mathbf{m}$

Calculate the mass flux. You can copy and paste the symbol of the
diffusivity. (You can also obtain it with Numlock on, using [Alt]**0208**).

10-3 Plate Viscometer   5 min

The force on the moving plate
of a viscometer is:

$$F = \eta \cdot v \cdot \frac{A}{h}$$

The parameters are given below. Calculate the viscosity and check that you
get the correct unit: $kg \cdot m^{-1} \cdot s^{-1}$

force	$F := 0.03 \cdot N$
velocity	$v := 1 \cdot cm \cdot s^{-1}$
plate area	$A := 6 \cdot cm^2$
liquid thickness	$h := 1 \cdot mm$

Viscosity is often given in the units $N \cdot s \cdot m^{-2}$
Type these in the final placeholder of your result to see whether Mathcad
accepts this result.

10-4 Temperature of a Spray Dryer   5 min

The outlet temperature of a laboratory spray dryer is governed by an energy
balance:

$$Q' = m' \cdot c_P \cdot \left(T_{out} - T_{in} \right)$$

(This is approximate: I have neglected a few terms.) The following values
are given:

heat flow	$Q' := 100 \cdot W$
mass flow of air	$m' := 0.1 \cdot kg \cdot min^{-1}$
specific heat of air	$c_P := 1000 \cdot J \cdot kg^{-1} \cdot K^{-1}$
inlet temperature (in K)	$T_{in} := (25 + 273) \cdot K$

Calculate the outlet temperature in K. Then replace the unit with the degree
Celsius from Math > Units.

10-5 Volume of an Ideal Gas   5 min

The volume of an ideal gas is governed by the ideal gas law:

$$p \cdot V = N \cdot R \cdot T$$

The following values are given:

(molar) amount	$N := 3 \cdot mol$
pressure	$p := 2 \cdot 10^5 \ Pa$

gas constant	$R = 8.314 \ J \cdot mol^{-1} \cdot K^{-1}$	Mathcad constant
temperature in K	$T := (25 + 273.16) \cdot K$	

Calculate the volume.

10-6 Falling Particle   5 min

A free falling particle has the following parameters:

|		
mass	$m := 10^{-3} \cdot kg$	
acceleration	$g = 9.807 \ m \cdot s^{-2}$	Mathcad constant
time after release	$t := 1.5 \cdot s$	

The following equations are given:

force on particle	$F = m \cdot g$	
velocity	$v = g \cdot t$	
distance fallen	$s = \dfrac{1}{2} \cdot g \cdot t^2$	Calculate these values.

10-7 Flow from a Vessel   10 min

Consider a vessel filled with water, open at the top and
with a small hole at the bottom. The velocity of the water
jet issuing from the hole is given by the Bernoulli equation:

$$\frac{v^2}{2} = g \cdot h$$
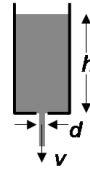
The jet has a diameter of about 0.8 times the diameter of the hole, so the
cross section has an area:

$$A = \frac{\pi}{4} \cdot (0.8 \cdot d)^2$$

Calculate the mass flow of water leaving:

$$m' = \rho \cdot v \cdot A$$

The parameters have the values:

height of water	$h := 0.4 \cdot m$
hole diameter	$d := 1 \cdot cm$
density of water	$\rho := 1000 \cdot \dfrac{kg}{m^3}$

Calculate the velocity, area and mass flow. Note that there is a problem:
you will have to redefine one of the units. Why is this? When you have
finished, consider the following. A more complete form of the Bernoulli
equation is:

$$\frac{v^2}{2} + g \cdot h + \frac{p}{\rho} = constant$$

The terms are often regarded as 'energies per mass'. So you would expect
the terms to have units of J/kg. However, the dimensions of the term do
not always look like that. For example:

$$g \cdot h = 0.004 \ N$$

Delete the unit expression and replace it by J/kg, then click outside the
expression. Mathcad should accept your changes, showing that the two
sets of units are identical.

Now make a mistake on purpose. Replace the units in the equation above
by the incorrect **J*kg**. Then click outside the expression. Mathcad will
have corrected your units!

10-8 Concentration in a Mixture   20 min

A mixture consists of water and inulin (a polysaccharide):

	water (W)	inulin (I)
mass percentage	$w_W := 90\%$	$w_I := 10\%$
'partial' density	$\rho_W := 1000 \cdot \dfrac{kg}{m^3}$	$\rho_I := 1400 \cdot \dfrac{kg}{m^3}$
molar mass	$M_W := 18 \cdot 10^{-3} \cdot \dfrac{kg}{mol}$	$M_I := 400 \cdot 10^{-3} \cdot \dfrac{kg}{mol}$

Consider one kilogram of water and calculate:
(1) the masses, (molar) amounts and volumes of water and inulin
(2) the total volume and the molar concentration of water and inulin.

10-9 Vapour Pressure of Water   5 min

You can use units in functions and also in plots. As an example we will use the Clapeyron equation for the vapour pressure of water. This contains the following parameters:

atmospheric pressure $\qquad\qquad\qquad\qquad p_0 := 10^5 \cdot \mathrm{Pa}$

enthalpy of vaporisation $\qquad\qquad\quad \Delta H_{vap} := 44 \cdot 10^3 \ \mathrm{J} \cdot \mathrm{mol}^{-1}$

gas constant $\qquad\qquad\qquad\qquad\quad R = 8.314 \ \mathrm{J} \cdot \mathrm{mol}^{-1} \cdot \mathrm{K}^{-1}$

boiling point (in K!) $\qquad\qquad\qquad T_B := (273 + 100) \cdot \mathrm{K}$

The vapour pressure as a function of temperature is approximately:

$$p_{vap}(T) := p_0 \cdot \exp\left(-\left(\frac{\Delta H_{vap}}{R}\right) \cdot \left(\frac{1}{T} - \frac{1}{T_B}\right)\right)$$

Check the values at $\qquad\qquad T_1 := T_B \qquad\quad T_2 := (273 + 20) \cdot \mathrm{K}$

(You know the value at the boiling point - that at room temperature is about two percent of this value.) Make a plot of the vapour pressure against temperature. Puts units in the unit placeholders, and choose a logarithmic axis for the vapour pressure.