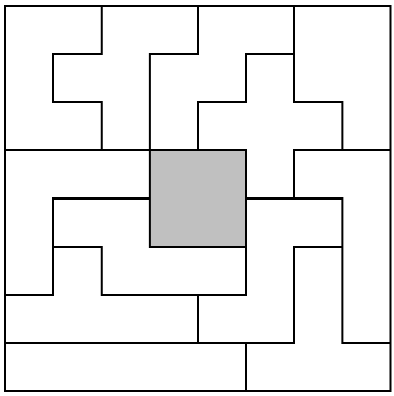
FACETS OF PENTOMINOES



In memory of Solomon W. Golomb (1932–2016),
the ‘founder’ of the pentominoes

# Preface

After the satisfying publication of our puzzle book *Exotische Sudoku's* with Brave New Books naturally the thought came up: why not compose another book on pentominoes?

Soon after I started collaborating with Aad van de Wetering on sudokus he introduced me to another field of amusement: pentominoes. My first interest was to tile the plane with copies of each of the twelve ones, rediscovering that it can always be done. Next I was acquainted with the fantastic website of Odette De Meulemeester (pentomino.classy.be) and her contests on it. Only with Wedstrijd 45, a pento-sudoku puzzle, I joined the scene. So Aad actually already had a long history behind him in the pentomino field. Some of his finest results have been collected in a separate chapter.

For a long time I liked to puzzle on 'ruitjespaper' (graph paper) and to play with a set of pentominoes (received from Odette), for example to make enclosures. Of course I also learned to use Aad vdW's magnificent program Poly3D (it can be downloaded freely from his website home.planet.nl/~avdw3b/). The many options of this program are staggering, as the name reveals it can even handle 3-dimensional polyominoes. Especially useful are the so-called 'must cubes', they facilitate greatly the search for e.g. enclosures. Actually the program cannot be praised enough. This book could never have been realized without it, not only because of its solving capabilities, almost every page benefits from its graphics: all figures have been made with Poly3D making this book very bright and colourful.

In collaborating with Aad vdW my main input was to come up with ideas. Often they called upon Aad's programming skills. Especially when I came up with the idea of ConMagic squares I tested his limits. But not only his, those of Helmut Postl, another excellent programmer, as well. As the saying goes: one fool can pose more problems than ten programmers can solve.

From the onset the aspiration was to compose a book in *five* parts – for obvious reasons – covering the main facets of pentominoes:

Part 1: congruences, Part 2: enclosures, Part 3: tilings, Part 4: ConMagic squares and Part 5: miscellaneous, a mixture of special subjects.

They will be explained in the Introduction. It was often difficult to make choices: what examples to show, what details to mention explicitly and at the same time to avoid the trap of repeating too many well-known results. Hopefully the reader will find many new ideas, points of view and results in this book. We do not claim intellectual property of them. As much as possible due credit is given by name to previous ideas and results. Especially Pieter Torbijn (1924–2007) deserves to be mentioned, he previously had a close collaboration with Aad vdW.

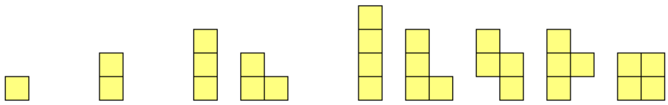
I would like to point at one pretty idea that came up during the composition of this book: that of transforming one congruence into another one with the same pentominoes by applying a reflection. The idea initially turned up in Chapter 1.6 and became more clear in the extra Chapter 1.8. Further in the later inserted Chapter 4.3 it returns. Obviously the book was not written in one go!

We dedicate this book to Odette who more than anyone else has made an effort through her many travels to 'pentomize' this planet bringing the joy of playing with pentominoes to young and old. To put her in the spot-light we close this book with a fairy tale in which she is starring.

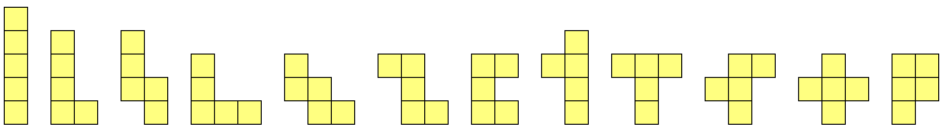
Aad Thoen                                        Amsterdam, July 2018
thoenaad37@gmail.com

## Introduction

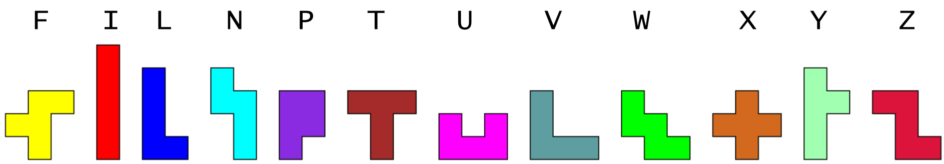
A polyomino of order $n$ ($n$ = 1, 2, 3, …) consists of $n$ 1×1 squares joined along the sides into any shape which is allowed to contain holes. In the language of polyominoes – ‘poly’ means ‘many’ – the counting from one to ten goes like this: mono, do, tro, tetro, pento, hexo, hepto, octo, nono, deco. There are one monomino, one domino, two trominoes and five tetrominoes:



The protagonists – or heroes if you like – of this book are the pentominoes, there are already twelve of them as there are twelve ways of joining five squares into one shape ignoring rotations and/or reflections:



The usual way to depict them is shown below with their characteristic colours and in alphabetical order. Due to their appearance – to see this you sometimes need to rotate them – they are named after twelve letters from the alphabet:




As can be witnessed from Odette’s website tremendously many fanciful things can be done with the pentominoes. It would be tempting to copy for example the many beautiful results (Records) of the 51 *wedstrijden* (contests). We will mention only a few of them. Apart from some general stuff we will concentrate on our investigations into congruences, enclosures, tilings and more specific subjects. They are pointed out briefly below.

However before we proceed we need to mention an important ingredient, that of symmetry. If something abstract can be done symmetrically it is experienced as more beautiful, in particular if it can be done rectangularly. So first we explain the different kinds of symmetry.

*Symmetry*

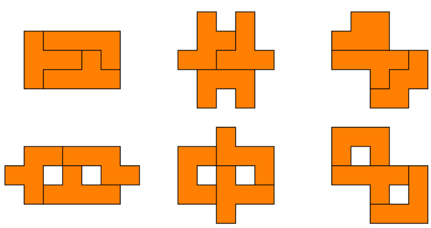
The playground for our pentominoes is the square grid and so the symmetries of the figures we build from them correspond with the symmetries of the square. These form a so-called *group* of eight elements, the dihedral group D4 = {1, 2, 3, 4, 5, 6, 7, 8}. We can describe its elements in terms of permutations of the four corners of the square:

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A | B | D | A | C | D | B | C | D | C | B | A | A | D | C | B |
| D | C | C | B | B | A | A | D | A | B | C | D | B | C | D | A |
| (1) | | (2) | | (3) | | (4) | | (5) | | (6) | | (7) | | (8) | |

The first four elements form the subgroup of rotations: (1) the identity, (2) rotation about 90°, (3) rotation about 180° which is the same as point-symmetry, (4) rotation about 270° = –90°. The other elements are reflections: (5) reflection in the horizontal axis, (6) reflection in the vertical axis, (7) reflection in the diagonal AC, (8) reflection in the diagonal BD.

We can identify eight kinds of symmetry of polyomino-figures, they correspond to the subgroups of D4. (a) no symmetry, {1}; (b) rotational symmetry, {1 – 4}; (c) point-symmetry, {1, 3}; (d) symmetry in one orthogonal axis, {1, 5} or {1, 6}; (e) symmetry in one diagonal axis, {1, 7} or {1, 8}; (f) symmetry in two orthogonal axes, {1, 3, 5, 6}; (g) symmetry in two diagonal axes, {1, 3, 7, 8}; (h) complete symmetry, {1 – 8}. The pentominoes themselves can serve as examples. X has complete symmetry: it has rotational symmetry and all the mirror symmetries. I has symmetry in two orthogonal axes. T and U have symmetry in one orthogonal axis. V and W have symmetry in one diagonal axis. Z has point-symmetry. The remaining ones (F, L, N, P, Y) have no symmetry.

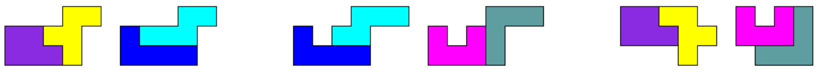
In Chapter 5.7 you can find the symmetrical figures consisting of two different pentominoes (organized into SymMagic squares). These cannot be point-symmetric. Below we give six figures from three pentominoes which are not only point-symmetric but even symmetric in two axes.



*The doubly-symmetric figures consisting of three pentominoes.*

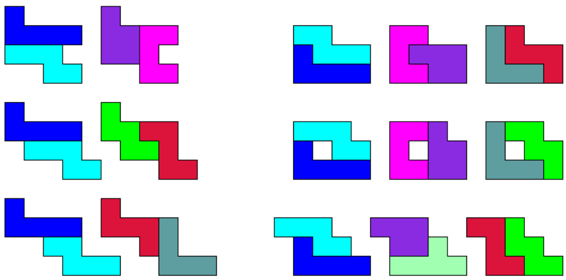
*Congruences*

It is a bit like chemistry to build larger polyominoes from the pentominoes by 'gluing' them along the sides. Our domain however is dominated by the occurrence of congruence. Two pentominoes A and B can be glued in several different ways. When possibly the result yields the same shape as gluing C and D we say that AB is congruent with CD and we notate this relation by a simple dash: AB-CD. We call this a 22 (two-two) congruence. The congruence relation however is not an equivalence relation, that is, AB-CD and CD-EF do not imply AB-EF. We give an example:



FP-LN *and* LN-UV *but* FP-UV *appears to be impossible.*

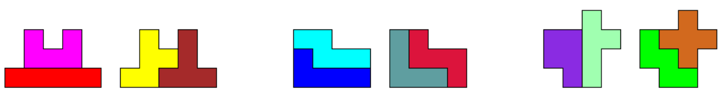
Six more congruences with LN follow below, those on the right are 222 (two-two-two) congruences. Note the multi-congruency: a congruence AB-CD can be realized in different ways.



*Three 22 resp. three 222 congruences.*

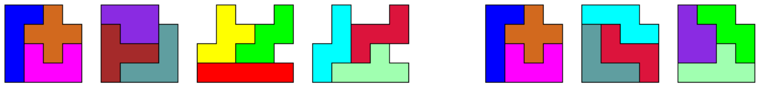
There appear to be no 2222 congruences, that is, no four pairs AB can be congruent to each other in which all the pentominoes are different. Although we also consider AA-BB and AB-BC congruences we generally demand that the pentominoes in the (combination of) congruence(s) are different.

Next, on the other hand there are 22+22 and even 22+22+22 congruences, in the last ones a complete set of pentominoes is consumed.



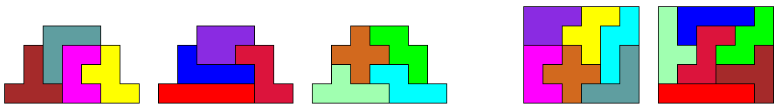
*One of the eighteen 22+22+22 congruences.*

Going up from 2 to 3 pentominoes we can likewise show 33 (three-three), 33+33 and 333 congruences. There are no 3333 congruences.



*A 33+33 resp. 333 congruence.*

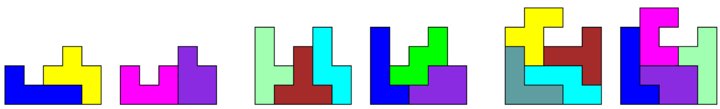
Like 22+22+22 and 33+33 two more congruences that use a complete set of pentominoes are the 444 and 66 congruences.



*A symmetric 444 resp. rectangular 66 congruence.*

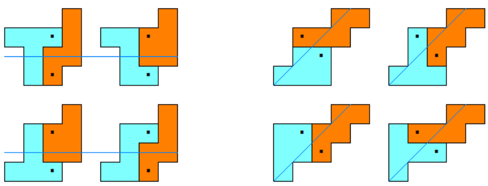
We did not investigate these ‘high’ congruences. The reader can find a host of 444 congruences in “20 problem” (Congruence section) on Odette’s website as well as a host of 33+33 congruences in “2x2 problem”.

A particular kind of *nn* congruences is shown below. Can the reader recognize the property that is shared by them? A hint is given in the figure caption.



*So-called ‘rectangly’ 22, 33 resp. 44 congruences.*

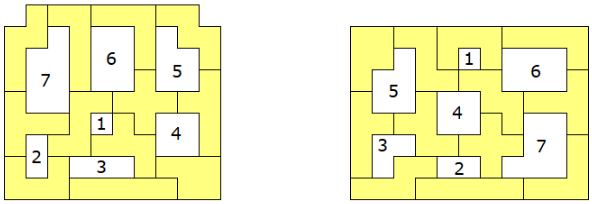
Finally we show simultaneous pairs of congruences AB-CD + AD-CB which can be transformed into each other by a reflection. Note that below the blue pentomino plus orange square with dot are congruent on the left and right; these parts are reflected to give another congruence.



*Reflection in the axis transforms congruence AB-CD into AD-CB.*

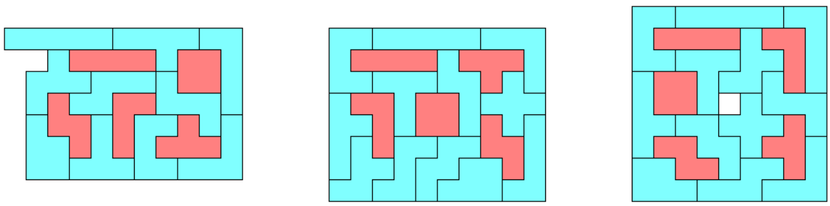
*Enclosure*

With the pentominoes (some of them or a complete set) we can enclose empty spaces or holes. Below twice a complete set encloses holes of sizes 1 through 7. Also here the rule holds: if it can be done rectangularly the better.




*Two enclosures of spaces of size 1 through 7.*

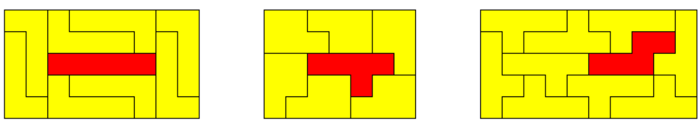
To enclose a full set of tetrominoes by ten different pentominoes in a 7×10 rectangle fails sadly. One can view the enclosure as a sea and the tetrominoes as islands.



*Three enclosures of a tetromino set.*

In Part 2 we start with enclosing eyes (1×1 squares) and end with enclosing a complete set of pentomino-islands for which we will need two pentomino sets.

Worth mentioning is an alternative way of enclosing: enclosure by copies of one and the same pentomino. In that way three pentominoes (L, P and Y) can rectangularly enclose each of the pentominoes. We show three easy examples, for enclosing I, T or W by Y's you need 10×15 rectangles!

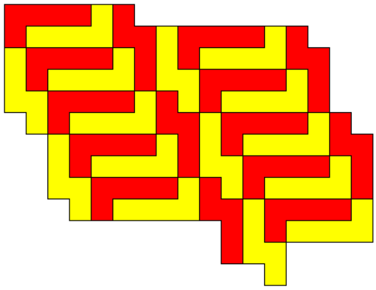


*Enclosure of* I, Y *resp.* N *by* L's, P's *resp.* Y's.

A variation on the above is given in Potpourri 65 (Enclosed pentomino in a congruent figure) on Odette's website.

*Tilings*

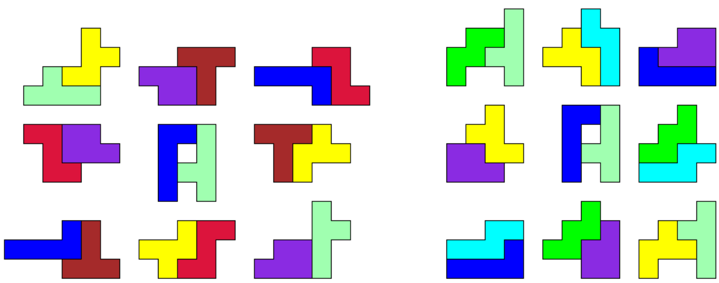
Well known are the tilings of the plane by triangles, squares and hexagons. In Part 3 we explore tilings of the plane by one pentomino or combinations of them. Special attention is given to so-called black-and-white tilings, these tilings can be coloured by using just two colours. In the example below these colours are yellow and red, it is clear how the pattern extends.



*BW-tiling of the plane by* L *and* Y.

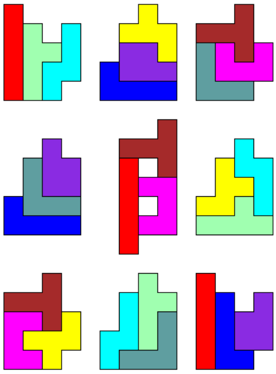
*ConMagic squares*

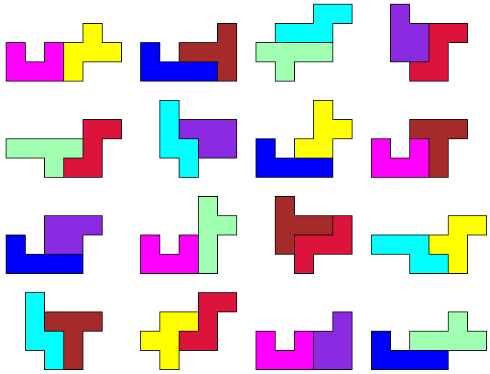
In Part 4 we return to the congruences but then organized into orthogonal Latin squares. Each row or column should contain the same set of pentominoes and the contents of each cell should be different, certain congruence demands are imposed on them. It all started with 3×3 squares with two pentominoes per cell with the demand that cells in point-symmetric position should be congruent, see Chapter 4.1. It was previously published in the newsletter of *Cubism For Fun* (CFF 91, July 2013, pp. 18–22). The centre plays no role in these squares as it cannot be point-symmetric. Below though we used the centre to depict the letter A twice making a pun on the initials AT and AW!



*Two 3×3 ConMagic squares depicting AT and AW.*

Next the number of pentominoes per cell can be increased. In the example left below even all mid cells are congruent. Due to the congruences NY-LP and UV-FU the pentominoes I and T could be chosen freely to make the centre point-symmetric. Also the order of the square can be increased (right below), still with the demand that point-symmetric cells are congruent.



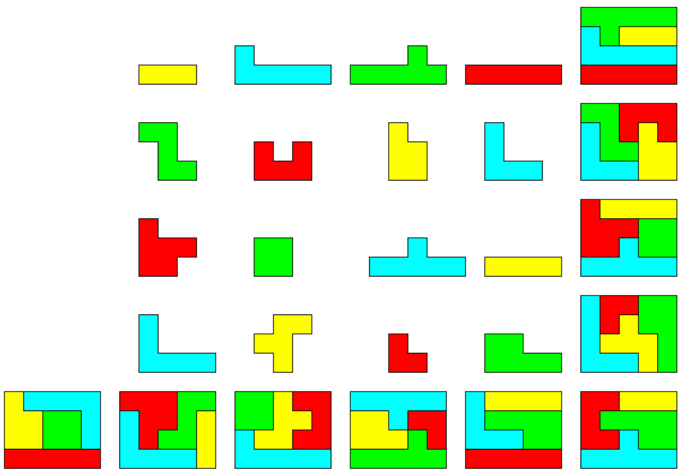


*A 3×3 (with 3 pentominoes per cell) resp. 4×4 ConMagic square.*

Other congruence demands can be imposed, for example line congruency: the cells in a row or column should be congruent, see Chapters 4.4 and 4.5. More fanciful are side and corner congruencies, see Chapter 4.6.

*Miscellaneous*

Chapters 5.1 and 5.2 are devoted to GeoMagic squares, a beautiful invention of Lee Sallows. They are a geometric generalization of the (numerical) magic square, numbers are replaced by geometric shapes.

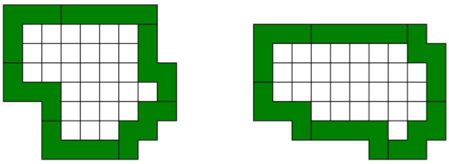


*A 4×4 GeoMagic square.*

In the example the so-called geosum is a 4×5 rectangle. Can you find a 3×3 GeoMagic square with only (different) pentominoes?

We competed in two exciting contests, one on Yin-Yang congruences and the other on axis-symmetric enclosures of maximal spaces. A selection of its results are rendered in Chapters 5.3 and 5.4. More special enclosures, multiple ones and point-symmetric ones, follow in Chapter 5.5.

A funny challenge is linking six string-like pentominoes into a closed chain such that the remaining six pentominoes are enclosed (Chapter 5.6). There appear to be two instances which each admit five solutions to the enclosed area.



*The enclosed areas can in 5 ways be filled with the remaining 6 pentominoes.*

As already said Chapter 5.7 is devoted to symmetric figures consisting of two different pentominoes. The symmetry property is used to construct Latin squares which we name SymMagic.

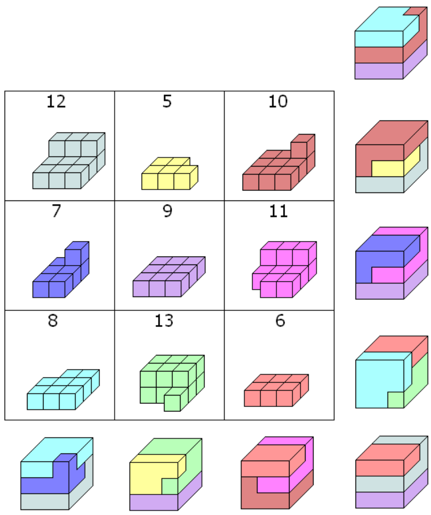
Finally, a very different subject is that of shift-figures, see Chapter 5.8.

NB: to save breadth we will abbreviate pentomino(es) to pento(s). As a matter of fact Odette likes to call our heroes ‘pentokes’ meaning ‘little pentos’.

## Honours of Aad van de Wetering

In this chapter some of the successes of Aad van de Wetering are gathered, successes of which he is most proud.

*GeoMagic square with polycubes*

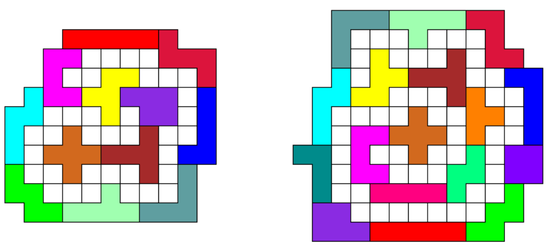



The so-called geosum of the above GeoMagic square is a 3×3×3 cube. The square nicely demonstrates the 3D option of Poly3D.

*Narrow path*

This is a kind of enclosure problem. The challenge is to enclose with the pentos a as long as possible path of width one square, obviously the path may be quite winding. It is quite a feat that Aad managed to enclose a path of 36 squares.

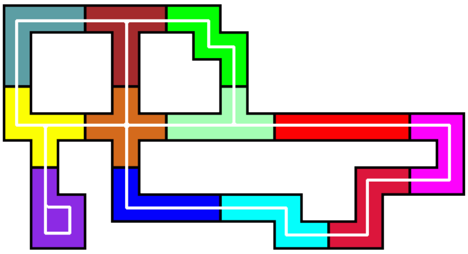
A variation on this theme is to make a longer path by adding the five tetros in the enclosure. It was the subject of Wedstrijd 26 on Odette's website, the contest was dedicated to Pieter Torbijn.



*Narrow paths of 36 resp. 52 squares enclosed by pentos resp. +tetros.*

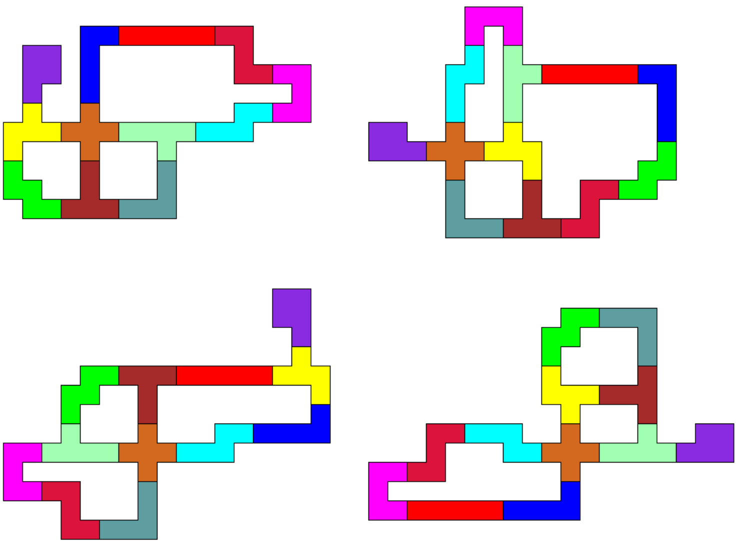
*Pentomino railways*

In a very specific way one can build from the pentos a railway. To start with one needs to draw tracks on the pentos and the pentos can only be connected at the 'ends' in such a way that the tracks are prolonged. To make this clear an example follows:



The X is a genuine crossing, the F, T and Y are a kind of T-junctions, the P acts as a turning station while the remaining pentos are just connectors.

Aad found no less than 69 essentially different solutions (e.g. switching V and W or reversing N make no difference). They can all be found on Aad's website where they are ordered according to the enveloping rectangle. Just four more fine examples are shown below. For the origins of the problem refer to Wedstrijd 21 on Odette's website.

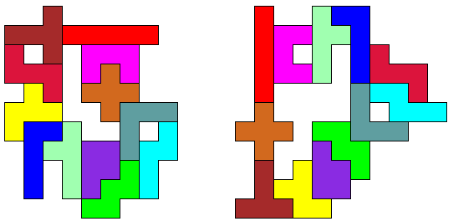


*Four more railways.*

*Symmetric chains*

The task for Wedstrijd 37 on Odette's website stems from Stefano Popovski: place the pentos into a closed chain ABCD…L such that the consecutive pairs AB, BC, CD,…,KL as well as LA are all symmetric in some axis.

This fine symmetry problem was not only solved completely by Aad but also by Bob Henderson and Peter Jeuken. Three of the 21 solutions are shown below: `FZTIUXVNWPYL`, `FTXIUYLZNVWP` and `FZLIUTYPNWVX`. The reader will have no problem in finding the axes of symmetry.

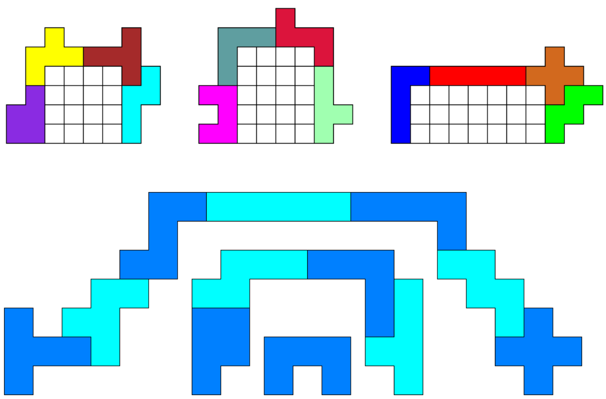


*Three symmetry chains.*

*Three bridges*

Living in the village of Driebruggen, literally meaning Threebridges, Aad was destined to come up with Wedstrijd 4: build three bridges from a pento set (four pentos per bridge) such that the bridged spaces are rectangles, maximize the total area of the three rectangles.

Many solutions with the maximum of 57 squares were sent in, the one by Livio Zucca is shown. An unrelated illustration of Driebruggen by Aad follows next, it is alas not completely symmetric.

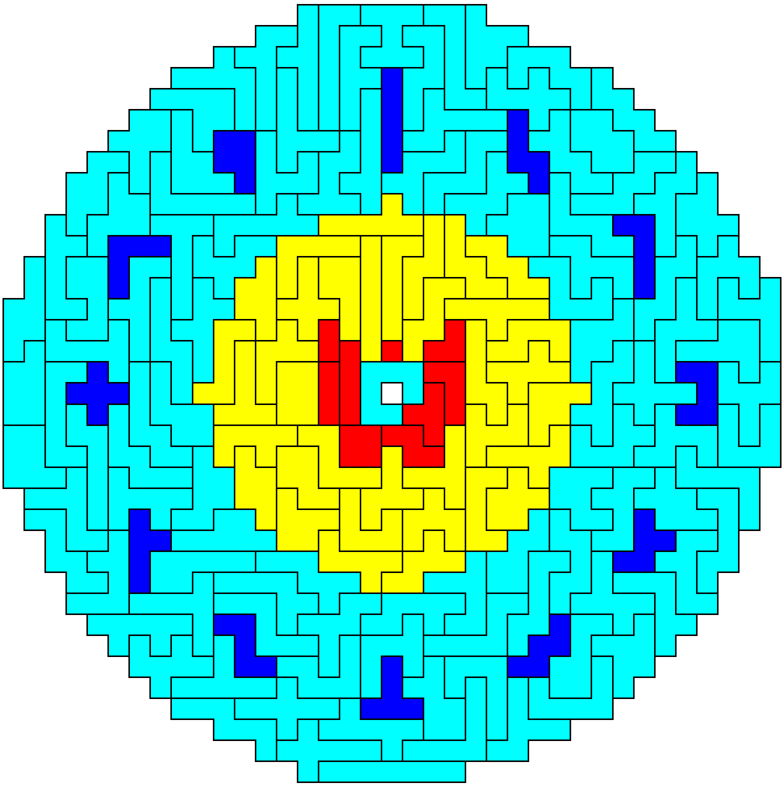


*'Threebridges' in two ways.*

*Polyomino clock*

This chapter is closed with a more serious feat. Aad managed to build a clock from all polyominoes of order 1 through 7.

In red the smallest polyominoes seem to depict a W. The twelve pentos of course represent the twelve hours of the clock. In yellow are the hexominoes (35 in total) and the outer ring is filled with the heptominoes with the exception of the unique one containing a hole which is placed in the centre (there is a total of 108 heptominoes). Quite remarkable!



*Polyomino clock.*