QUESTIONS AND ANSWERS INTRODUCTION TO OPTIMIZATION FOR ECONOMISTS

Questions and Answers Introduction to Optimization for Economists

Jacob Engwerda

1st edition

2026

OPEN PRESS TiU

Open Press Tilburg University
Tilburg, The Netherlands

Design and typesetting: LINE UP boek en media bv
Cover design: Bas Ekkers

ISBN: 9789403866727
DOI: https://doi.org/10.56675/9789403866727

Contents

Prologue

The book *An Introduction to Optimization for Economists* originates from a crashcourse *refresher in mathematics* that I taught for many years at Tilburg University to research master and starting PhD students economics, prior to my retirement in 2019.
After my retirement, the lecture notes from this course were used as the foundation for writing this book. Additional theory was inorporated, and to make the material accessible at different levels, numerous exercises were added.

In this accompanying volume to the book, answers to all exercises are provided. A disclaimer applies here. Many exercises have not been checked yet by students. Though I did my best in elaborating them accurately, probably still typo's or other inaccuracies will be there. Readers are encouraged to report them on the site mentioned below, so that other users may benefit from this later on.

My personal hope is that students around the world will enhance their understanding through the mathematical approach to optimization presented in this book. In particular, I believe the book can serve as a useful stepping stone towards the study of more specialized areas in optimization.

I would like to thank Tilburg University, and especially Tilburg University Press, for making it possible to publish this book as open access. I am particularly grateful to Beatriz Lourenço Barrocas Neves Ferreira for her support throughout the publishing process.
Last but not least, I would like to thank my wife, Carine, for giving me the space and freedom to devote so much of my time writing this book and conducting research.

If you have any comments on the book, you are most welcome to post and share them on the DOI-site: https://doi.org/10.56675/9789403860336.

Writing this book has been a rewarding way to spend part of my retirement. If you feel inclined to show your appreciation, you may consider a small donation to the bank account below. I will use this money then to make a hiking tour with Carine. You may use next Wise bank account for that purpose: Jacob Christiaan Engwerda, IBAN: BE21967500746803, Belgium, Swift/BIC: TRWIBEB1XXX.

Hoping you may benefit from the book, Jacob Engwerda.

Preface

Economics is a social science that studies the production, distribution, and consumption of goods and services. A central concern of the discipline is how economic agents behave and interact, and how economies function as a whole. In line with this focus, a fundamental distinction in economics is made between microeconomics and macroeconomics.

Microeconomics analyzes the behavior of individual components of the economy, such as households and firms, buyers and sellers, as well as markets and their interactions. Macroeconomics, in contrast, examines the economy as a whole and addresses issues such as unemployment, inflation, economic growth, and monetary and fiscal policy.

To support such analyses, economists frequently rely on mathematical tools to model specific economic situations. By using mathematical abstractions of real-world phenomena, one aims to gain deeper insight into the underlying mechanisms at work. Ultimately, this approach is intended to improve the allocation of scarce resources.

In the book *An Introduction to Optimization for Economists*, we review a number of standard mathematical tools that are used to determine optima of functions. Accordingly, the focus lies on the second aspect of the analysis described above: given a specific mathematical model of an economic problem, how can one derive conclusions that—provided the model captures the essential features of the problem—lead to a better understanding of agent behavior and a more efficient allocation of resources.

Beginning with the concept of a function, the main text presents conditions under which optima can be derived. Both single-variable and multivariable functions are considered. Since resources are typically scarce, we then study conditions for finding optima when the domain of a function is restricted, leading to so-called constrained optimization problems. A special and important subclass of these problems consists of convex optimization problems. Owing to their favorable numerical properties—particularly in the case of linear programming—they are widely used in applications and are therefore treated in a separate section. Finally, we consider optimization problems in which the outcome is influenced by the actions of more than one agent.

This book is intended for students who wish to learn mathematical techniques that they can later apply to their own economic problems. The main text is accessible to students in economics as well as those pursuing more technical fields of study. Mathematical technicalities are deliberately kept to a minimum. The goal is to provide students with a basic understanding of the underlying ideas and an intuitive sense of the conditions under which specific mathematical tools can be applied and how they are used.

Each chapter concludes with a set of exercises, which are divided into three categories. The first category consists of exercises that allow students to test their understanding of the presented material. The second category encourages deeper engagement with mathematical rigor, including proofs of theorems introduced in the main text and related topics. The third category aims to illustrate how the theoretical tools can be applied in economics to gain better insight into real-world phenomena.

In this accompanying volume, we provide solutions to all exercises in the book.

Notation and Symbols

$f(x) = O(x)$ as $x \to 0$	there exist $c,\ \epsilon$ such that $\|f(x)\| \leq c\|x\|$ when $0 < \|x\| \leq \epsilon$.
$\mathbb{R}$	Real numbers
$\mathbb{R}_+$	strictly positive real numbers
$\mathbb{R}^n$	set of vectors with n real entries
$\mathbb{R}^{n \times m}$	set of $n \times m$ matrices with real entries
$f^{-1}(x)$	inverse function of $f(x)$
$f^{(n)}(x)$	n^{th} order derivative of $f(x)$
x^T	transpose of the vector x
$\|x\|$	length $\sqrt{x^T x}$ of vector x, or, distance vector x to the origin
$\frac{\partial f}{\partial x_i}$, or, $D_i f(x)$, or $f^{'}_{x_i}(x)$	partial derivative of f w.r.t. x_i
$D_u f(x)$	directional derivative $f(x)$ in the direction u
$Df(x) = [D_1 f(x) \cdots D_n(x)]$	row matrix of partial derivatives (or, Jacobian of $f(x)$)
$f^{'}(x)$	the derivative of $f(x)$ (which equals $Df(x)$ if it exists)
C^1	set of differentiable functions which partial derivatives are continuous
C^k	set of functions which all k^{th}-order partial derivatives are continuous
$f''(x)$, or, Hessian $H(x)$	second order derivative of $f(x)$, i.e. $[(D_1^{'} f(x))^T \cdots (D_n^{'} f(x))^T]^T$
A^T	transpose of matrix A
$A > 0$	positive definite matrix A
$A \geq 0$	positive semi-definite matrix A
$A < 0$	negative definite matrix A
$A \leq 0$	negative semi-definite matrix A
det A	determinant of matrix A
A^{-1}	inverse of matrix A
$I_{n \times n}$	identity matrix in $\mathbb{R}^{n \times n}$
$T_S(a)$	tangent cone to S at a
$TL_S(a)$	set of linearized feasible directions to S at a

$\subset$	subset
$\cup$	union
$\cap$	intersection
$\forall$	for all
$\exists$	there exists
$\mathbb{R}^n/\{0\}$	all $x \in \mathbb{R}^n$ except vector 0
$x \geq y$	componentwise inequality between vectors x and y
$f_D(\lambda)$	function associated with dual problem
$\mathcal{A}$	unit simplex
$S\backslash T$	all elements from S not belonging to T
$\bar{x}_{-i}$	$\bar{x}$, except entry i which is deleted
$\bar{x}_{\backslash i}$	$\bar{x}$, except entry i which is arbitrary
$\mathcal{R}(x_L)$	best response set follower against x_L
Λ	nonnegative orthant

CHAPTER 1

Static Optimization: The Scalar Case

1.1 Functions of one variable

EXERCISE 1.1 Determine for the rules f, defined below, a set D as large as possible such that f defines a function from $D \to \mathbb{R}$. Calculate the range of the corresponding function.

a. $f(x) = x + 1.$ b. $f(x) = \frac{1}{x}.$ c. $f(x) = \frac{1}{(x-1)^2}.$ d. $f(x) = \{x \mid x^2 = y,$ where $y \in [1,4]\}.$

ANSWER 1.1

a. $D = \mathbb{R}$; $f(D) = \mathbb{R}$.

b. $D = \mathbb{R} \setminus \{0\}$; $f(D) = D$.

c. $D = \mathbb{R} \setminus \{1\}$; $f(D) = \{y \mid y > 0\}$.

d. $D = [1,2]$; $f(D) = [1,4]$.

EXERCISE 1.2 Determine which of the functions f, defined below, is invertible. For all functions that are invertible, determine the inverse function f^{-1}.

a. $f(x) = x + 1.$ b. $f(x) = x^2,\ x \in [-1,1].$ c. $f(x) = -x^2 + 1,\ x \in [0,1].$

ANSWER 1.2

a. invertible; $y = x + 1 \to x = y - 1 \to f^{-1}(x) = x - 1.$

b. not invertible.

c. invertible: $y = -x^2 + 1 \to x^2 = 1 - y \to x = \sqrt{1-y} \to f^{-1}(x) = \sqrt{1-x}.$

EXERCISE 1.3

a. Simplify $\frac{{}^a\log(x)}{{}^a\log(y) + {}^a\log(z)}.$

b. Simplify $e^{2\ln(x)} - \ln(e^{2x}).$

c. Solve $2^{x^2} = 4^{-3x-4}.$

d. Solve $\left(\frac{1}{4}\right)^{x^2} = 64\left(2^{-2x}\right)^4.$

e. Solve $2\ln(x - \frac{1}{2}e) - \ln(x - \frac{3}{4}e) = 1.$

ANSWER 1.3

a. $\frac{{}^a\log(x)}{{}^a\log(yz)}$.

b. $e^{2\ln(x)} - \ln(e^{2x}) = e^{\ln(x^2)} - \ln(e^{2x}) = x^2 - 2x$.

c. $2^{x^2} = 4^{-3x-4} \to 2^{x^2} = 2^{2(-3x-4)} \to x^2 = 2(-3x-4) \to x^2 + 6x + 8 = 0 \to x = -2,\ x = -4$.

d. $\left(\frac{1}{4}\right)^{x^2} = 64\left(2^{-2x}\right)^4 \to 2^{-2x^2} = 2^6 2^{-8x} \to 2^{-2x^2} = 2^{6-8x} \to -2x^2 = 6 - 8x \to x = 1,\ x = 3.$

e. $2\ln(x - \frac{1}{2}e) - \ln(x - \frac{3}{4}e) = 1 \to \ln(x - \frac{1}{2}e)^2 - \ln(x - \frac{3}{4}e) = \ln(e) \to \ln\frac{(x-\frac{1}{2}e)^2}{x-\frac{3}{4}e} = \ln(e) \to \frac{(x-\frac{1}{2}e)^2}{x-\frac{3}{4}e} = e \to (x - \frac{1}{2}e)^2 = e(x - \frac{3}{4}e) \to x^2 - 2ex + e^2 = 0 \to x = e.$

EXERCISE 1.4 Let $f(x) = x^2 + 1$ and $g(x) = \log(x^2 + 1)$. Determine

a. $(f + g)(x)$.
b. $(fg)(x)$.
c. $\left(\frac{f}{g}\right)(x)$.
d. $(f \circ g)(x)$.
e. $(g \circ f)(x)$.

ANSWER 1.4

a. $(f + g)(x) = x^2 + 1 + \log(x^2 + 1)$.

b. $(fg)(x) = (x^2 + 1)\log(x^2 + 1)$.

c. $\left(\frac{f}{g}\right)(x) = \frac{x^2+1}{\log(x^2+1)}$.

d. $(f \circ g)(x) = \log^2(x^2 + 1) + 1$.

e. $(g \circ f)(x) = \log((x^2 + 1)^2 + 1)$.

1.2 Compact Sets

EXERCISE 1.5 Determine the boundary and accumulation points of next sets. Motivate which of these sets is bounded, open, closed or compact, respectively.

a. $[1, 2] \cup [3, \infty)$.
b. $\{1, 2\}$.
c. $[1, 2) \cup (2, 3]$.
d. $[1, 3) \cap (0, 2]$.

ANSWER 1.5

a. Boundary points: 1,2,3. Accumulation points: 1,2,3. Not bounded, not open, closed, not compact.

b. Boundary points: 1,2. Accumulation points: none. Bounded, not open, closed, compact.

c. Boundary points: 1,3. Accumulation points: 1,2,3. Bounded, not open, not closed, not compact.

d. $[1, 3) \cap (0, 2] = [1, 2]$ → Boundary points: 1,2. Accumulation points: 1,2. Bounded, not open, closed, compact.

1.3 Continuous Functions

EXERCISE 1.6 Motivate which of the next functions is continuous.

a. $f(x) = \begin{cases} \log(x^2 + 1), & x \le 0 \\ x, & x > 0 \end{cases}$.

b. $f(x) = xe^{x^2 + \ln(x^2+1)}$.

c. $f(x) = \begin{cases} \frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$.

ANSWER 1.6

a. $g(x) = x^2 + 1 \geq 1$ is continuous; $h(y) = \log(y)$ is continuous $\rightarrow h(g(x))$ is continuous; $k(x) = x$ is continuous. Notice $k(0) = h(g(0)) \rightarrow f$ is continuous.

b. $g(x) = x^2 + 1 \geq 1$ is continuous; $h(y) = \ln(y)$ is continuous $\rightarrow h(g(x))$ is continuous $\rightarrow z(x) = x^2 + h(g(x))$ is continuous; $k(y) = e^y$ is continuous $\rightarrow f(x) = xk(z(x))$ is continuous.

c. Let $g(x) = \frac{1}{x}$, $x \neq 0$. Consider the sequence $t_k = \frac{1}{k}$. Then, t_k converges to zero if k goes to infinity. Furthermore, $g(t_k) = k$ goes to infinity. So, $g(t_k)$ does not converge to zero. So, g is not continuous.

EXERCISE 1.7 Motivate whether the next functions have an optimum.

a. $f(x) = \frac{1}{x}$, $x > 0$.

b. $f(x) = \frac{x^2 - e^x}{\ln(x^2+2)}$, $-10 \leq x \leq 12$.

ANSWER 1.7

a. The range of f is $(0, \infty)$. Furthermore $f(x) < f(y)$ if $x > y$. So, f has no optimum.

b. $D := [-10, 12]$ is compact. f is continuous on D. So, by Weierstrass' theorem, f has both a global maximum and a global minimum on D.

1.4 Differentiable Functions

EXERCISE 1.8 Determine the derivative of the next functions $f(x)$.

a. $f(x) = x^2 + {}^2\log(x)$, $x > 0$.

b. $f(x) = \frac{2}{x^5} + 2^x$, $x \neq 0$.

c. $f(x) = \frac{\ln(x)}{x+1}$, $x > 0$.

d. $f(x) = x^2 e^{x^2} + 1$.

e. $f(x) = \log(x^2+1) + \sqrt{x^4+1}$.

ANSWER 1.8

a. $f^{'}(x) = 2x + \frac{1}{\ln 2}\frac{1}{x}$.

b. $f^{'}(x) = -\frac{10}{x^6} + 2^x \ln 2$.

c. $f^{'}(x) = \frac{\frac{1}{x}(x+1) - \ln(x)}{(x+1)^2}$.

d. $f^{'}(x) = 2xe^{x^2} + 2x^3 e^{x^2}$.

e. $f^{'}(x) = \frac{1}{\ln 10}\frac{2x}{x^2+1} + \frac{2x^3}{\sqrt{x^4+1}}$.

EXERCISE 1.9 Motivate which of the next functions is differentiable.

a. $f(x) = \frac{x^2 + e^x}{\ln(x^2+2)}$.

b. $f(x) = \begin{cases} x, & x \leq 0 \\ x+1, & x > 0 \end{cases}$.

c. $f(x) = x + |x|$.

d. $f(x) = \begin{cases} x, & x \leq 0 \\ 2x, & x > 0 \end{cases}$.

e. $f(x) = x|x| + \log(e^x + 4)$.

ANSWER 1.9

a. $g(x) = x^2 + 2 \geq 2$ is differentiable; $h(y) = \ln(y)$ is differentiable $\rightarrow h(g(x))$ is differentiable, moreover $h(g(x)) \neq 0$; $k(x) = x^2 + e^x$ is differentiable $\rightarrow f = \frac{k(x)}{h(g(x))}$ is differentiable.

b. Let $g(x) := x$ and $h(x) := x + 1$. Then $g(0) = 0 \neq 1 = h(0)$. So, f is not continuous at $x = 0$. Therefore f is not differentiable at $x = 0$.

c. $f(x) = 2x$ if $x \geq 0$ and $f(x) = 0$ if $x \leq 0$. So, the tangent line to the graph of f at $x = 0$ is not uniquely determined. Therefore, f is not differentiable at $x = 0$.

d. Similar to item c. it follows that the tangent line to the graph of f at $x = 0$ is not uniquely determined. Therefore, f is not differentiable at $x = 0$.

e. $f(x) = x^2 + \log(e^x + 4)$ is differentiable if $x > 0$ and $f(x) = -x^2 + \log(e^x + 4)$ is differentiable if $x < 0$. Let $g_+(x) = x^2$ and $g_-(x) = -x^2$. Then, both g_+ and g_- are differentiable at $x = 0$ with derivative 0 $\rightarrow h(x) = x|x|$ is differentiable at $x = 0$. Also, $\log(e^x + 4)$ is differentiable at $x = 0 \rightarrow$ f is differentiable at $x = 0$ too.

1.5 First-Order Criterium for an Optimum of a Scalar Function

EXERCISE 1.10 Determine the stationary points of the next functions.

a. $f(x) = xe^x$. b. $f(x) = \frac{x}{x^2+1}$. c. $f(x) = \ln(x^2 + 2x + 10)$. d. $f(x) = \frac{e^{\frac{1}{2}x^2}}{x}$, $x \neq 0$.

ANSWER 1.10

a. $f'(x) = (x + 1)e^x = 0 \rightarrow x = -1$.

b. $f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = 0 \rightarrow x^2 = 1 \rightarrow x = 1,\ x = -1$

c. $f'(x) = \frac{2x+2}{x^2+2x+10} = 0 \rightarrow x = -1$.

d. $f'(x) = \frac{x^2 e^{\frac{1}{2}x^2} - e^{\frac{1}{2}x^2}}{x^2} = 0 \rightarrow x^2 = 1 \rightarrow x = 1,\ x = -1$

1.6 Monotonicity of Functions

EXERCISE 1.11 Determine the intervals where the next functions $f(x)$ are decreasing and increasing, respectively.

a. $f(x) = x^3 - 3x$. b. $f(x) = e^{x^2}$. c. $f(x) = \frac{1}{x^2}$, $x \neq 0$. d. $f(x) = \frac{e^x}{x^2}$, $x \neq 0$.

ANSWER 1.11

a. $f'(x) = 3x^2 - 3 = 0 \rightarrow x = 1,\ x = -1$.

```
  +     -      +  f'(x)
-----0------0-----
    -1      1      x
```

$\rightarrow f$ is monotonically increasing on $(-\infty, -1] \cup [1, \infty)$;

f is monotonically decreasing on $[-1, 1]$.

b. $f'(x) = 2xe^{x^2} = 0 \rightarrow x = 0$.

```
  -        +    f'(x)
-------0--------
       0         x
```

$\rightarrow f$ is monotonically decreasing on $(-\infty, 0]$;
f is monotonically increasing on $[0, \infty)$.

c. $f^{'}(x) = \frac{-2}{x^3},\ x \neq 0.$

```
  +        -    f'(x)
-------*--------
       0         x
```

$\rightarrow f$ is monotonically increasing on $(-\infty, 0)$;
f is monotonically decreasing on $(0, \infty)$.

d. $f^{'}(x) = \frac{x^2e^x - 2xe^x}{x^4} = \frac{x(x-2)e^x}{x^4} = 0, \rightarrow x = 2.$

```
  +        -       +   f'(x)
-------*-------0-------
       0       2        x
```

$\rightarrow f$ is monotonically increasing on $(-\infty, 0) \cup [2, \infty)$;
f is monotonically decreasing on $(0, 2]$.

EXERCISE 1.12 Determine the optima of the next functions, using Theorem 1.34.

a. $f(x) = x^3 - 3x^2 + 3x + 1.$

b. $f(x) = x^7 - \frac{7}{5}x^5.$

c. $f(x) = \begin{cases} -(x+1)^2, & x \leq -1; \\ 0, & x \in (-1, 0); \\ x^2, & x \geq 0. \end{cases}$

ANSWER 1.12

a. $f^{'}(x) = 3x^2 - 6x + 3 = 0 \rightarrow x = 1.$

```
  +        +    f'(x)
-------0-------
       1         x
```

$\rightarrow f$ has a saddle point at $x = 1$.

b. $f^{'}(x) = 7x^6 - 7x^4 = 0 \rightarrow 7x^4(x^2-1) = 0 \rightarrow x = -1,\ x = 0,\ x = 1.$

```
  +        -       -      +   f'(x)
-------0-------0-------0------
      -1       0       1       x
```

$\rightarrow f$ has a maximum at $x = -1$, a saddle point at $x = 0$ and a minimum at $x = 1$. As $f(100.000) > f(-1)$, the maximum is not global; as $f(-100.000) < f(1)$, the minimum is not global.

c. Notice that f is differentiable for all $x \neq \{-1, 0\}$. Furthermore, $g(x) = -(x+1)^2$ is differentiable at $x = -1$ with $g^{'}(-1) = 0$ and $h(x) = 0$ is differentiable at $x = -1$ with $h^{'}(-1) = 0$. As $g(-1) = h(-1)$ it follows that f is differentiable at $x = -1$ too. Similarly it follows that f is also differentiable at $x = 0$.
$\rightarrow f^{'}(x) = -2(x+1),\ x \leq -1$; $f^{'}(x) = 0,\ x \in (0, 1)$ and $f^{'}(x) = 2x,\ x \geq 0$.

```
  +        0       +   f'(x)
-------0-------0-------
      -1       0        x
```

$\rightarrow f$ has a local maximum at $x \in [-1, 0)$ and a local minimum at $x \in (-1, 0]$.

EXERCISE 1.13 Determine the optima of the next functions $f(x)$. Indicate whether the optima are local or global optima.

a. $f(x) = x^3 - 3x + 1$.

b. $f(x) = 3x^4 - 4x^3 + 5$.

c. $f(x) = xe^{-\frac{1}{2}x^2}$, $-2 \leq x \leq 2$.

d. $f(x) = \frac{x}{1+x^2}$.

ANSWER 1.13

a. $f^{'}(x) = 3x^2 - 3 \rightarrow$ Stationary points: $x = 1,\ x = -1$.

```
  +        -       +   f'(x)
-------0-------0-------
      -1       1        x
```

$\rightarrow f$ has a local maximum at $x = -1$ and a local minimum at $x = 1$. As $f(100.000) > f(-1)$, the maximum is not global; as $f(-100.000) < f(1)$, the minimum is not global.

b. $f^{'}(x) = 12x^3 - 12x^2 = 12x^2(x-1) \rightarrow$ Stationary points: $x = 0,\ x = 1$.

```
  -        -       +   f'(x)
-------0-------0-------
       0       1        x
```

$\rightarrow f$ has a saddle point at $x = 0$ and a global minimum at $x = 1$.

c. $f^{'}(x) = e^{-\frac{1}{2}x^2} - x^2e^{-\frac{1}{2}x^2} = (1-x^2)e^{-\frac{1}{2}x^2} \rightarrow$ Stationary points: $x = 1,\ x = -1$.

```
    -       +       -   f'(x)
|-------0-------0-------|
-2     -1       1       2  x
```

$\rightarrow f$ has a local minimum at $x = -1$ and $x = 2$; a local maximum at $x = -2$ and $x = 1$.
Note that the domain is compact and f is continuous. Therefore, by Weierstrass' theorem, f attains both a global maximum and a global minimum. Since $f(-1) < f(2)$, f has a global minimum at $x = -1$. Similarly, since $f(-2) < f(1)$, f has a global maximum at $x = 1$.

d. $f^{'}(x) = \frac{1-x^2}{(1+x^2)^2} = 0 \rightarrow x^2 = 1 \rightarrow x = 1,\ x = -1$

```
  -       +         -  f'(x)
------0-------0-------
     -1       1         x
```

$\rightarrow f$ has a local minimum at $x = -1$ and a local maximum at $x = 1$. Note $f(x) \rightarrow 0$ if $x \rightarrow \pm\infty$, $f(1) > 0$ and $f(-1) < 0$. Consequently, f has a global minimum at $x = -1$ and a global maximum at $x = 1$.

1.7 Convexity and Second-Order Criteria for an Optimum of a Scalar Function

EXERCISE 1.14 Indicate the intervals where the next functions are convex and concave, respectively.
a. $f(x) = e^x$. b. $f(x) = \ln(x),\ x > 0$. c. $f(x) = e^{-\frac{1}{2}x^2}$. d. $f(x) = e^x - \ln(x),\ x > 0$.

ANSWER 1.14
a. $f^{'}(x) = e^x \rightarrow f”(x) = e^x$. $f”(x) > 0 \rightarrow f$ is convex on $(-\infty, \infty)$.
b. $f^{'}(x) = \frac{1}{x} \rightarrow f”(x) = \frac{-1}{x^2}$. $f”(x) < 0 \rightarrow f$ is concave on $(0, \infty)$.
c. $f^{'}(x) = -xe^{-\frac{1}{2}x^2} \rightarrow f”(x) = (x^2-1)e^{-\frac{1}{2}x^2} \rightarrow f”(x) = 0$ at $x = 1$ and $x = -1$

```
  +       -         +  f"(x)
------0-------0-------
     -1       1         x
```

$\rightarrow f$ is convex on $(-\infty, -1] \cup [1, \infty)$ and f is concave on $[-1, 1]$.
d. $f^{'}(x) = e^x - \frac{1}{x} \rightarrow f”(x) = e^x + \frac{1}{x^2} \rightarrow f”(x) > 0 \rightarrow f$ is convex on $(0, \infty)$.

EXERCISE 1.15 Verify the correctness of the result stated in Remark 1.44 by checking the advertized conditions for the examples considered in Example 1.43.

ANSWER 1.15
a. $f(x) = x^4 \rightarrow f^{'}(0) = f”(0) = f^{(3)}(0) = 0$; $f^{(4)}(0) = 24 > 0 \rightarrow$ item i. of Remark 1.44 applies $\rightarrow f$ has local minimum at $x = 0$.
b. $f(x) = -x^4 \rightarrow f^{'}(0) = f”(0) = f^{(3)}(0) = 0$; $f^{(4)}(0) = -24 < 0 \rightarrow$ item ii. of Remark 1.44 applies $\rightarrow f$ has local maximum at $x = 0$.
c. $f(x) = x^3 \rightarrow f^{'}(0) = f”(0) = 0$; $f^{(3)}(0) = 6 \neq 0 \rightarrow$ item iii. of Remark 1.44 applies $\rightarrow f$ has saddle point at $x = 0$.

EXERCISE 1.16 Consider the function $f(x) = 1 + (x^2 - 1)^3$. Determine all extremal points of f and indicate whether they are local or global optima.

ANSWER 1.16

$f^{'}(x) = 6x(x^2-1)^2 \rightarrow$ Stationary points: $x=0$, $x=1$ and $x=-1$.

```
 -       +       -     +   f'(x)
-----0-------0-------0------
    -1       0       1     x
```

$\rightarrow f$ has a local minimum at $x=-1$ and $x=1$; a local maximum at $x=0$. $f(-1)=f(1) \rightarrow f$ has a global minimum at $x=-1$ and $x=1$; $f(100) > f(0) \rightarrow f$ has no global maximum.

EXERCISE 1.17 Show that the function $f(x) = x^8 - 4x^2$ has a local maximum at $x=0$. Is this also a global maximum? Determine all extremal points of f on $[-1.1, 1.1]$.

ANSWER 1.17

a. $f^{'}(x) = 8x^7 - 8x \rightarrow f^{''}(x) = 56x^6 - 8 \rightarrow f^{'}(0) = 0$ and $\rightarrow f^{''}(0) = -8 < 0 \rightarrow f$ has local maximum at $x=0$.

   $f(100) > f(0) \rightarrow f$ has not a global maximum at $x=0$.

b. $f^{'}(x) = 8x(x^6-1) \rightarrow$ stationary points at $x=0$, $x=-1$ and $x=1$

```
     -       +       -      +   f'(x)
  |------0-------0-------0------|
-1.1    -1       0       1    1.1  x
```

   $\rightarrow f$ has a local minimum at $x=-1$ and $x=1$; a local maximum at $x=0$. $f(-1.1) = f(1.1) > f(0)$ $\rightarrow f$ has a global maximum at $x=-1.1$ and $x=1.1$; $f(-1) = f(1) \rightarrow f$ has global minimum at $x=1$ and $x=-1$.

1.8 Sexi Functions

EXERCISE 1.18 Consider the function $f(x) = x^2 \sin\frac{1}{x}$, $x \neq 0$. Let $f(0) = 0$.

a. Use the definition of differentiability to show that $f(x)$ is differentiable at $x=0$.

b. Show that $f^{'}(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$, $x \neq 0$.

c. Show that $f^{'}(x)$ is not continuous at $x=0$.

ANSWER 1.18

a. From the definition of differentiability it follows that we have to show that next difference quotient has a limit if Δx approaches zero.

$$\frac{f(0+\Delta x) - f(0)}{\Delta x} = \frac{(\Delta x)^2 \sin\frac{1}{\Delta x} - 0}{\Delta x} = \Delta x \sin\frac{1}{\Delta x}.$$

   Note that $|\Delta x \sin\frac{1}{\Delta x}| \leq |\Delta x|$. Therefore, $|\Delta x \sin\frac{1}{\Delta x}|$ approaches zero if $|\Delta x|$ approaches zero.

b. For $x \neq 0$, f is differentiable. Using the standard calculation rules we have $f^{'}(x) = 2x \sin \frac{1}{x} + x^2 \frac{-1}{x^2} \cos \frac{1}{x}$.

c. Note that if x approaches zero $2x \sin \frac{1}{x}$ approaches zero (see item a.). On the other hand $\cos \frac{1}{x}$ fluctuates between -1 and 1 if x approaches zero. So, $f^{'}(x)$ does not approach a limiting value if x approaches zero. Therefore $f^{'}(x)$ is not continuous at $x = 0$.

Mixed Exercises

EXERCISE 1.19 Determine for the functions f below all stationary points. Indicate whether f attains at these points a local or global minimum/maximum.

a. $f(x) = x^2 - 6x + 8$.
b. $f(x) = x^4 - 2x^2 + 2$.
c. $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x - 1$.
d. $f(x) = x^5 + 1$.
e. $f(x) = \frac{x}{e^x}$.
f. $f(x) = \frac{4x+3}{x^2+1}$.
g. $f(x) = \frac{e^x}{1+x^2}$.
h. $f(x) = \frac{2x^2+4}{\frac{1}{4}x^4+\frac{1}{2}x^2+1}$.

ANSWER 1.19

a. $f^{'}(x) = 2x - 6 \rightarrow$ Stationary point: $x = 3$.

```
     -           +    f'(x)
-----------0----------
           3          x
```

$\rightarrow f$ has a global minimum at $x = 3$.

b. $f^{'}(x) = 4x^3 - 4x = 4x(x^2 - 1) \rightarrow$ Stationary points: $x = 0$, $x = 1$ and $x = -1$.

```
  -     +      -      +  f'(x)
-----0------0------0-----
    -1      0      1     x
```

$\rightarrow f$ has a local minimum at $x = -1$ and $x = 1$; f has a local maximum at $x = 0$.
$f(100) > f(0) \rightarrow f$ has no global maximum.
$f(-1) = f(1) \rightarrow f$ has global minimum at $x = -1$ and $x = 1$.

c. $f^{'}(x) = x^2 - x - 2 = (x - 2)(x + 1) \rightarrow$ Stationary points: $x = -1$ and $x = 2$.

```
  +        -        +  f'(x)
-----0-----------0-----
    -1           2     x
```

$\rightarrow f$ has a local maximum at $x = -1$; f has a local minimum at $x = 2$.
$f(100) > f(-1) \rightarrow f$ has no global maximum.
$f(-100) < f(2) \rightarrow f$ has no global minimum.

d. $f^{'}(x) = 5x^4 \rightarrow$ Stationary point: $x = 0$.

```
  +         -   f'(x)
--------0-------
        0       x
```

$\rightarrow f$ has a global maximum at $x = 0$.

e. $f^{'}(x) = \frac{e^x - xe^x}{e^{2x}} \rightarrow$ Stationary point: $x = 1$.

```
  +       -    f'(x)
-------0------
       1       x
```

$\rightarrow f$ has a global maximum at $x = 1$.

f. $f^{'}(x) = \frac{4(x^2+1)-2x(4x+3)}{(x^2+1)^2} = \frac{-4x^2-6x+4}{(x^2+1)^2} \rightarrow$ Stationary points: $x = -2$ and $x = \frac{1}{2}$.

```
  -      +        -   f'(x)
-----0--------0-------
    -2       0.5      x
```

$\rightarrow f$ has a local minimum at $x = -2$; f has a local maximum at $x = \frac{1}{2}$.

Note $f(x) \rightarrow 0$ if $x \rightarrow \pm\infty$, $f(-2) < 0$ and $f(\frac{1}{2}) > 0$. Consequently, f has a global minimum at $x = -2$ and a global maximum at $x = \frac{1}{2}$.

g. $f^{'}(x) = \frac{e^x(x^2+1)-2xe^x}{(x^2+1)^2} = \frac{(x^2-2x+1)e^x}{(x^2+1)^2} \rightarrow$ Stationary point: $x = 1$.

```
    +          +    f'(x)
---------0----------
         1          x
```

$\rightarrow f$ has a saddle point at $x = 1$.

h. $f^{'}(x) = \frac{4x(\frac{1}{4}x^4+\frac{1}{2}x^2+1)-(2x^2+4)(x^3+x)}{(\frac{1}{4}x^4+\frac{1}{2}x^2+1)^2} = \frac{-x^3(x^2+4)}{(\frac{1}{4}x^4+\frac{1}{2}x^2+1)^2} \rightarrow$ Stationary point: $x = 0$

```
     +            -    f'(x)
-----------0-----------
           0           x
```

$\rightarrow f$ has a global maximum at $x = 0$.

EXERCISE 1.20 Determine for the functions g below all stationary points. Indicate whether g attains at these points a local or global minimum/maximum.

a. $g(x) = \log(1 + x^2)$. b. $g(x) = e^{x^2+2x}$. c. $g(x) = \sqrt{\frac{1}{4}x^4 + x^2 + 1}$. d. $g(x) = xe^{-\frac{1}{2}x^2}$.

ANSWER 1.20

a. $g^{'}(x) = \frac{2x}{\ln(10)(1+x^2)} \rightarrow$ Stationary point: $x = 0$.

```
     -           +    g'(x)
-----------0----------
           0          x
```

$\rightarrow g$ has a global minimum at $x = 0$.

b. $g^{'}(x) = (2x+2)e^{x^2+2x} \rightarrow$ Stationary point: $x = -1$.

```
     -           +    g'(x)
-----------0----------
          -1          x
```

$\rightarrow g$ has a global minimum at $x = 0$.

c. $g^{'}(x) = \frac{x^3+2x}{2\sqrt{\frac{1}{4}x^4+x^2+1}} = \frac{x(x^2+2)}{2\sqrt{\frac{1}{4}x^4+x^2+1}} \rightarrow$ Stationary point: $x = 0$

```
     -           +    g'(x)
-----------0----------
           0          x
```

$\rightarrow g$ has a global minimum at $x = 0$.

d. $g^{'}(x) = e^{-\frac{1}{2}x^2} - x^2e^{-\frac{1}{2}x^2} = (1-x^2)e^{-\frac{1}{2}x^2} \rightarrow$ Stationary points: $x = -1$ and $x = 1$.

```
 -         +       -  g'(x)
-----0-----------0-----
    -1           1     x
```

$\rightarrow g$ has a local minimum at $x = -1$; g has a local maximum at $x = 1$.

Note $g(x) \rightarrow 0$ if $x \rightarrow \pm\infty$, $g(1) > 0$ and $g(-1) < 0$. Consequently, g has a global minimum at $x = -1$ and a global maximum at $x = 1$.

EXERCISE 1.21 Determine for the functions h below all stationary points. Indicate whether h attains at these points a local or global minimum/maximum.

a. $h(x) = (e^{x^2}+1)^2$. b. $h(x) = \frac{x\sqrt{x}}{e^{-\frac{3}{2}x+1}}$, $x \geq 0$. c. $h(x) = (x^2+1)^{\frac{1}{3}}(x^2+2)^{\frac{2}{3}}$.

d. $h(x) = \ln(\frac{1}{1+x^2}) + \ln(\sqrt{1+x^2})$.

ANSWER 1.21

a. $h^{'}(x) = 4x(e^{x^2}+1)e^{x^2} \rightarrow$ Stationary point: $x = 0$.

```
     -           +    g'(x)
-----------0----------
           0          x
```

$\rightarrow h$ has a global minimum at $x = 0$.

b. $h^{'}(x) = \frac{\frac{3}{2}x^{\frac{1}{2}}e^{-\frac{3}{2}x+1}+\frac{3}{2}e^{-\frac{3}{2}x+1}x^{\frac{3}{2}}}{(e^{-\frac{3}{2}x+1})^2} = \frac{\frac{3}{2}x^{\frac{1}{2}}e^{-\frac{3}{2}x+1}(1+x)}{(e^{-\frac{3}{2}x+1})^2} \rightarrow$ No stationary points

```
          +         g'(x)
|---------------------
0                     x
```

$\rightarrow h$ has a global minimum at $x = 0$.

c. $h^{'}(x) = \frac{1}{3}(x^2+1)^{-\frac{2}{3}}2x(x^2+2)^{\frac{2}{3}} + (x^2+1)^{\frac{1}{3}}\frac{2}{3}(x^2+2)^{-\frac{1}{3}}2x$
$= (x^2+1)^{-\frac{2}{3}}(x^2+2)^{-\frac{1}{3}}2x(\frac{1}{3}(x^2+2)+\frac{2}{3}(x^2+1)) \rightarrow$ Stationary point: $x = 0$.

```
     -           +    g'(x)
-----------0----------
           0          x
```

$\rightarrow h$ has a global minimum at $x = 0$.

d. $h^{'}(x) = -\frac{2x}{1+x^2} + \frac{x}{1+x^2} = -\frac{x}{1+x^2} \rightarrow$ Stationary point: $x = 0$.

```
     +           -    g'(x)
-----------0----------
           0          x
```

$\rightarrow h$ has a global maximum at $x = 0$.

EXERCISE 1.22 Determine for the functions f below all extremal points. Indicate whether f attains at these points a local or global minimum/maximum.

a. $f(x) = x^2 - 4x + 3,\ x \in [0, 5]$.
b. $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x,\ x \in [0, 4]$.
c. $f(x) = \frac{1}{3}x^3 - x + 1,\ x \in [-2, 2)$.
d. $f(x) = \ln(1 + x),\ x \in [0, 10]$.
e. $f(x) = \frac{x^2}{1-x},\ x \in [\frac{3}{2}, 5]$.
f. $f(x) = \frac{x^2}{1+x},\ x \in [-4, -1) \cup (-1, 4]$.
g. $f(x) = \frac{1}{1+x^2},\ x \in [-1, \infty)$.
h. $f(x) = \frac{x}{1+x^2},\ x \in [-\frac{1}{2}, \infty)$.

ANSWER 1.22

a. $f^{'}(x) = 2x - 4 \rightarrow$ Stationary point: $x = 2$.

```
     -           +        g'(x)
|-----------0----------|
0           2          5   x
```

$\rightarrow f$ has a global minimum at $x = 2$.
$f(0) < f(5) \rightarrow f$ has a global maximum at $x = 5$ and a local maximum at $x = 0$.

b. $f^{'}(x) = x^2 - 4x + 3 \rightarrow$ Stationary points: $x = 2$.

```
   +       -      +       g'(x)
|-----0--------0-----|
0     1        3     4   x
```

$\rightarrow f$ has a local maximum at $x = 1$ and $x = 4$; f has a local minimum at $x = 0$ and $x = 3$.
$f(1) = f(4) \rightarrow f$ has a global maximum at $x = 1$ and $x = 4$;
$f(0) = f(3) \rightarrow f$ has a global minimum at $x = 0$ and $x = 3$.

c. $f^{'}(x) = x^2 - 1 \rightarrow$ Stationary points: $x = 1$ and $x = -1$.

```
   +       -      +       g'(x)
|------0--------0-----)
-2     -1       1     2   x
```

$\rightarrow f$ has a local maximum at $x = -1$; f has a local minimum at $x = -2$ and $x = 1$.

$f(-1) = f(2) \rightarrow f$ has a global maximum at $x = -1$;

$f(-2) = f(1) \rightarrow f$ has a global minimum at $x = -2$ and $x = 1$.

d. $f^{'}(x) = \frac{1}{1+x} \rightarrow$ No stationary points.

```
          +           g'(x)
|------------------|
0                   10 x
```

$\rightarrow f$ has a global minimum at $x = 0$; f has a global maximum at $x = 10$.

e. $f^{'}(x) = \frac{2x(1-x)+x^2}{(1-x)^2} = \frac{x(2-x)}{(1-x)^2} \rightarrow$ Stationary points: $x = 2$.

```
    +          -        g'(x)
 |------0-------------|
1.5     2             5  x
```

$\rightarrow f$ has a global maximum at $x = 2$; f has a local minimum at $x = \frac{3}{2}$ and $x = 5$.

$f(\frac{3}{2}) > f(5) \rightarrow f$ has a global minimum at $x = 5$.

f. $f^{'}(x) = \frac{2x(1+x)-x^2}{(1+x)^2} = \frac{x(2+x)}{(1+x)^2} \rightarrow$ Stationary points: $x = -2$ and $x = 0$.

```
    +   -       -     +   g'(x)
 |----0----)(-----0-------|
-4   -2    -1     0       4 x
```

$\rightarrow f$ has a local maximum at $x = -2$ and $x = 4$; f has a local minimum at $x = -4$ and $x = 0$.

$f(-1.01) < f(0)$ and $f(-1.01) < f(-4) \rightarrow f$ has no global minimum.

$f(-0.99) > f(-2)$ and $f(-0.99) > f(4) \rightarrow f$ has no global maximum.

g. $f^{'}(x) = -\frac{2x}{(1+x^2)^2} = \rightarrow$ Stationary points: $x = 0$.

```
    +          -        g'(x)
 |------0-------------
 -1     0               x
```

$\rightarrow f$ has a global maximum at $x = 0$; f has a local minimum at $x = -1$.

$f(5) < f(-1) \rightarrow f$ has no global minimum.

h. $f^{'}(x) = \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2} \rightarrow$ Stationary point: $x = 1$.

```
     +          -        g'(x)
  |------0-------------
-0.5     1               x
```

$\rightarrow f$ has a global maximum at $x = 1$; f has a local minimum at $x = -0.5$.

Note $f(x) \rightarrow 0$ if $x \rightarrow \infty$, $f(-0.5) < 0$. Consequently, f has a global minimum at $x = -0.5$.

EXERCISE 1.23 Solve the next equations.

a. $2 * 2^x + 4 * 2^x = 24.$

b. $2^{2x} + 2^{x+2} = 5.$

c. $\frac{1}{4} * 2^{2x} + 4^x + \frac{3}{4^{1-x}} = 32.$

d. $\ln(e^{x^2}) + e^{\ln(x)} = 2.$

e. $\ln(5x - 7) - \ln(x - 1) = \ln(x + 1),\ x > \frac{7}{5}.$

f. $\ln(x^2 - 2x + 1) = \frac{2}{\ln(x-1)},\ x > 1.$

ANSWER 1.23

a. $2 * 2^x + 4 * 2^x = 24 \Leftrightarrow 6 * 2^x = 24 \Leftrightarrow 2^x = 4 \Leftrightarrow 2^x = 2^2 \Leftrightarrow x = 2.$

b. $2^{2x} + 2^{x+2} = 5 \Leftrightarrow (2^x)^2 + 4 * 2^x - 5 = 0$, or, with $y := 2^x$, $y^2 + 4y - 5 = 0 \Leftrightarrow y = -5$ or $y = 1 \Leftrightarrow$ $2^x = -5$ or $2^x = 1 \Leftrightarrow x = 0.$

c. $\frac{1}{4} * 2^{2x} + 4^x + \frac{3}{4^{1-x}} = 32 \Leftrightarrow 4^{-1} * 4^x + 4^x + \frac{3}{4} * 4^x = 32 \Leftrightarrow 2 * 4^x = 32 \Leftrightarrow 4^x = 16 \Leftrightarrow x = 2$

d. $\ln(e^{x^2}) + e^{\ln(x)} = 2 \Leftrightarrow x^2 + x = 2 \Leftrightarrow x = 1$ or $x = -2.$

e. $\ln(5x - 7) - \ln(x - 1) = \ln(x + 1) \Leftrightarrow \ln \frac{5x-7}{x-1} = \ln(x + 1) \Leftrightarrow \frac{5x-7}{x-1} = x + 1 \Leftrightarrow 5x - 7 = x^2 - 1 \Leftrightarrow$ $x^2 - 5x + 6 = 0 \Leftrightarrow x = 3$ or $x = 2.$

f. $\ln(x^2 - 2x + 1) = \frac{2}{\ln(x-1)} \Leftrightarrow \ln((x - 1)^2) = \frac{2}{\ln(x-1)} \Leftrightarrow 2\ln(x - 1) = \frac{2}{\ln(x-1)} \Leftrightarrow$ with $y := \ln(x - 1)$, $2y^2 = 2 \Leftrightarrow y = 1$ or $y = -1 \Leftrightarrow \ln(x - 1) = 1$ or $\ln(x - 1) = -1 \Leftrightarrow x - 1 = e$ or $x - 1 = \frac{1}{e} \Leftrightarrow$ $x = 1 + e$ or $x = 1 + \frac{1}{e}$

EXERCISE 1.24 Determine the boundary and accumulation points of next sets. Motivate whether these sets are

bounded/open/closed or compact, respectively

a. $(-1, 2] \cup (3, 4).$

b. $[-1, 2) \cup (2, 4].$

c. $[-1, 2] \cup [3, 4].$

d. $(-1, 2) \cup (2, \infty).$

ANSWER 1.24

a. Boundary points: 2

Accumulation points: -1, 2, 3, 4

Set is bounded: $|x| < 10$ for all $x \in S$.

Set is not open: every interval $(2 - \epsilon_1, 2 + \epsilon_2)$ contains an element not belonging to S $(\epsilon_i > 0)$.

Set is not closed: every interval $(-1 - \epsilon_1, -1 + \epsilon_2)$ contains an element belonging to S $(\epsilon_i > 0)$. Or, $t_k = -1 + \frac{1}{k} \to -1$, with $t_k \in S$ for all $k > 1$, whereas its limit $-1 \notin S$.

Set is not compact: it is not closed.

b. Boundary points: -1, 4

Accumulation points: -1, 2, 4

Set is bounded: $|x| < 10$ for all $x \in S$.

Set is not open: every interval $(-1 - \epsilon_1, -1 + \epsilon_2)$ contains an element not belonging to S $(\epsilon_i > 0)$.

Set is not closed: every interval $(2 - \epsilon_1, 2 + \epsilon_2)$ contains an element belonging to S $(\epsilon_i > 0)$. Or, $t_k = 2 + \frac{1}{k} \to 2$, with $t_k \in S$ for all $k > 1$, whereas its limit $2 \notin S$.

Set is not compact: it is not closed.

c. Boundary points and accumulation points: -1, 2, 3, 4

Set is bounded: $|x| < 10$ for all $x \in S$.

Set is not open: every interval $(-1-\epsilon_1, -1+\epsilon_2)$ contains an element not belonging to S ($\epsilon_i > 0$).
Set is closed: complement set is open. Proof: construct appropriate intervals for the cases: $x < -1$, $2 < x < 3$ and $x > 4$, respectively.
Set is compact: it is both closed and bounded.

d. Boundary points: none
Accumulation points: -1, 2
Set is not bounded: Assume there exists an m such that $|x| < m$ for all $x \in S$. Choose $x = m + 1$, clearly $x \in S$ and $|x| > m$.
Set is open: construct an appropriate interval for the cases $-1 < x < 2$ and $x > 2$, respectively.
Set is not closed: every interval $(2-\epsilon_1, 2+\epsilon_2)$ contains an element belonging to S ($\epsilon_i > 0$). Or, $t_k = 2 + \frac{1}{k} \to 2$, with $t_k \in S$ for all $k > 1$, whereas its limit $2 \notin S$.
Set is not compact: it is neither closed nor bounded.

EXERCISE 1.25 Determine the largest domain for next functions g where they are continuous, respectively differentiable. Motivate your answer.

a. $g(x) = x + \ln(1+x^2)$.
b. $g(x) = |x+1| + e^{x-x^2}$.
c. $g(x) = \frac{\sqrt{x-1}}{x}$.
d. $g(x) = \frac{\ln(1+x)}{x}$.
e. $g(x) = \begin{cases} \frac{x^2-1}{x+1}, \ x \neq 1; \\ 0, \ x = 1. \end{cases}$
f. $g(x) = \begin{cases} x^2+1, \ x \leq 0; \\ |x+1|, \ x > 0. \end{cases}$

ANSWER 1.25

a. g is defined on $x \in \mathbb{R}$.
g is continuous on $x \in \mathbb{R}$ (both $g_1(x) = x$ and $g_2(x) = \ln(1+x^2)$ are continuous on this set).
g is differentiable on $x \in \mathbb{R}$ (both $g_1(x) = x$ and $g_2(x) = \ln(1+x^2)$ are differentiable on this set).

b. g is defined on $x \in \mathbb{R}$.
g is continuous on $x \in \mathbb{R}$ (both $g_1(x) = |x+1|$ and $g_2(x) = e^{x-x^2}$ are continuous on this set).
g is differentiable on $x \in \mathbb{R}\backslash\{-1\}$ ($g_1(x) = |x+1|$ is differentiable for all $x \neq -1$ and $g_2(x) = e^{x-x^2}$ is differentiable everywhere).

c. g is defined on $x \geq 1$.
g is continuous on $x \geq 1$ (both $g_1(x) = \sqrt{x-1}$ and $g_2(x) = \frac{1}{x}$ are continuous on this set).
g is differentiable on $x > 1$ ($g_1(x) = \sqrt{x-1}$ is differentiable for all $x > 1$ and $g_2(x) = \frac{1}{x}$ is differentiable for all $x \geq 0$).

d. g is defined on $D := \{x \mid x > -1, \ x \neq 0\}$.
g is continuous on D ($g_1(x) = \ln x + 1$ is continuous for all $x > -1$ and $g_2(x) = \frac{1}{x}$ is continuous for all $x \neq 0$).
g is differentiable on D ($g_1(x) = \ln x + 1$ is differentiable for all $x > -1$ and $g_2(x) = \frac{1}{x}$ is differentiable for all $x \neq 0$).

e. g is defined on $D := \{x \mid x \neq -1\}$.
Note that at $x = 1$, $g_1(x) := \frac{x^2-1}{x+1}$ equals 0. So, g is continuous on D.
g is differentiable on D, since g_1 is differentiable on D.

f. g is defined on $\mathbb{R}$.
Note that $g_1(x) := x^2 + 1$ is continuous for all $x \leq 0$ and $g_2(x) := |x+1| = x+1$, for all $x > 0$, is continuous. As $g_1(0) = g_2(0)$, g is continuous on $\mathbb{R}$.
g_1 is differentiable for all x, with $g_1^{'} = 2x$. g_2 is differentiable for all $x > -1$, with $g_2^{'} = 1$. So, $g_1^{'}(0) \neq g_2^{'}(0)$. Therefore, g is differentiable for all $x \neq 0$.

EXERCISE 1.26 Determine the intervals where the next functions h are decreasing and increasing, respectively.

a. $h(x) = x^2 - 4x + 4$.
b. $h(x) = -x^3 + 3x^2 - 3x + 1$.
c. $h(x) = x^4 - \frac{8}{3}x^3 + 2x^2 + 1$.
d. $h(x) = e^{-x^2+2x}$.
e. $h(x) = \ln(1+x),\ x > 0$.

ANSWER 1.26

a. $h^{'}(x) = 2x - 4 \rightarrow h^{'}(x) = 0$ at $x = 2$

```
   -          +    h'(x)

---------0---------

         2          x
```

$\rightarrow h$ is increasing on $[2, \infty)$ and decreasing on $(-\infty, 2]$.

b. $h^{'}(x) = -3x^2 + 6x - 3 = -3(x-1)^2 \rightarrow h^{'}(x) = 0$ at $x = 1$

```
   -          -    h'(x)

---------0---------

         1          x
```

$\rightarrow h$ is decreasing on $\mathbb{R}$.

c. $h^{'}(x) = 4x^3 - 8x^2 + 4x = 4x(x-1)^2 \rightarrow h^{'}(x) = 0$ at $x = 0$ and $x = 1$

```
  -      +      +    h'(x)

-----0-------0------

     0       1       x
```

$\rightarrow h$ is increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$.

d. $h^{'}(x) = (-2x + 2)e^{-x^2+2x} \rightarrow h^{'}(x) = 0$ at $x = 1$

```
   +          -    h'(x)

--------0---------

        1           x
```

$\rightarrow h$ is increasing on $(-\infty, 1]$ and decreasing on $[1, \infty)$.

e. $h^{'}(x) = \frac{1}{1+x} \rightarrow h^{'}(x) > 0$ for all $x > 0 \rightarrow h$ is increasing on $(0, \infty)$.

EXERCISE 1.27 Determine for the functions h in Exercise 1.26 the intervals where they are convex and concave, respectively.

ANSWER 1.27

a. $h^{"}(x) = 2 \rightarrow h$ is convex everywhere.

b. $h^{"}(x) = -6x + 6 \rightarrow h^{"}(x) = 0$ at $x = 1$.

```
   +          -    h"(x)
---------0---------
         1          x
```

$\rightarrow h$ is convex on $(-\infty, 1]$ and concave on $[1, \infty)$.

c. $h^{"}(x) = 12x^2 - 16x + 4 \rightarrow h^{"}(x) = 0$ at $x = \frac{1}{3}$ and $x = 1$

```
  +      -      +    h"(x)
-----0-------0------
    1/3      1       x
```

$\rightarrow h$ is convex on $(-\infty, \frac{1}{3}]$ and $[1, \infty)$ and concave on $[\frac{1}{3}, 1]$.

d. $h^{"}(x) = (-2 + (-2x + 2)^2)e^{-x^2+2x} = 2(2x^2 - 4x + 1)e^{-x^2+2x} \rightarrow h^{"}(x) = 0$ at $s = \frac{4-\sqrt{8}}{2}$ and $t = \frac{4+\sqrt{8}}{2}$

```
  +      -      +  h"(x)
-----0-------0-----
     s       t       x
```

$\rightarrow h$ is convex on $(-\infty, s]$ and $[t, \infty)$ and concave on $[s, t]$.

e. $h^{"}(x) = -\frac{1}{(1+x)^2} \rightarrow h^{"}(x) < 0$ for all $x > 0 \rightarrow h$ is concave for all $x > 0$.

Advanced, Deepening and Broadening Exercises

EXERCISE 1.28 Consider the intervals below in $\mathbb{R}$. Show that next statements are true.

a. The intervals (a, b), (a, ∞) and $(-\infty, b)$ are open.

b. The intervals $[a, b]$, $[a, \infty)$ and $(-\infty, b]$ are closed.

c. The intervals $[a, b)$ and $(a, b]$ are neither open or closed .

d. $\mathbb{R}$, is as well open as closed.

ANSWER 1.28

a. Let $x \in (a, b)$. Consider, e.g., $a_x = a$ and $b_x = b$. Then $(a_x, b_x) \subset (a, b)$.
Let $x \in (a, \infty)$. Consider, e.g., $a_x = a$ and $b_x = x + 1$. Then $(a, x, b_x) \subset (a, \infty)$.
Let $x \in (-\infty, b)$. Consider, e.g., $a_x = x - 1$ and $b_x = b$. Then $(a, x, b_x) \subset (-\infty, b)$.

b. The complement set of $[a, b]$ is $(-\infty, a) \cup (b, \infty)$, which is the union of two open sets (see item a.). By Proposition 1.10, the union of open sets is also open. A similar argument applies to the other intervals.

Another way to justify these statements is to show that the set of accumulation points of this interval coincides with its set of boundary points, namely $\{a, b\}$.

Consider the sequence $\{s_n\}_{n=1}^{\infty}$ defined by $s_n = a + \frac{b-a}{n}$. Then s_n converges to a, and both s_n and a lie in the interval $[a, b]$. Similarly, consider the sequence $\{t_n\}_{n=1}^{\infty}$ defined by $t_n = b + \frac{a-b}{n}$. Then t_n converges to b, and both t_n and b belong to $[a, b]$.

It is straightforward to verify that $[a, b]$ has no accumulation points other than $\{a, b\}$.

c. The interval $[a, b)$ is not open. Indeed, for any interval (a_x, b_x) containing a, there exists a point - such as $a - \frac{a-a_x}{2}$ - that lies in (a_x, b_x) but not in $[a, b)$.

To see that $[a, b)$ is not closed, consider its complement set $(-\infty, a) \cup [b, \infty)$. This set is not open; by an argument similar to the one above, one can show that $[b, \infty)$ is not open.

d. Let $x \in \mathbb{R}$. Define $a_x := x - 1$ and $b_x := x + 1$. Then $(a_x, b_x) \subset \mathbb{R}$, which shows that $\mathbb{R}$ is open. Moreover, for any sequence that converges to a limit s, we have $s \in \mathbb{R}$. Therefore, if $\mathbb{R}$ has any boundary points, they must lie in $\mathbb{R}$.

EXERCISE 1.29 Which of the next statements are true. Motivate your answer.

a. Every differentiable function is continuous.
b. There exist continuous functions which are not differentiable.
c. Every continuous function attains both a global maximum and a global minimum on $[0, 2]$.
d. There exist continuous functions that attain both a global maximum and a global minimum on $(0, 2)$.
e. A bounded set which contains all its boundary points is compact.
f. Every strictly increasing function is convex.
g. Every boundary point is an accumulation point.

ANSWER 1.29

a. True, see paragraph above Theorem 1.23.
b. True, see Figure 1.12.
c. True, $[0, 2]$ is compact, so by Weierstrass' Theorem 1.19 this result holds
d. True, consider $f(x) = x,\ 0 < x < 1,\ f(x) = -3x + 4,\ 1 \leq x < \frac{3}{2},\ f(x) = x - 2, \frac{3}{2} \leq x < 2$. Then f has a global maximum at $x = 1$ and a global minimum at $x = \frac{3}{2}$.
e. False, by definition a boundary point always belongs to the set. $S = (0, 2]$ has one boundary point (which belongs to S), but S is not closed (it does not contain all its accumulation points, i.e., 0 is an accumulation point of S but does not belong to S).
f. False, consider $h(x) = \ln(1 + x)$ for $x > 0$ (see Exercises 26e,27e).
g. True. If b is a boundary point, by definition $b \in S$. So, every interval (b_l, b_r) containing b, contains $b \in S$ and, again by definition of b being a boundary point, a point outside S. That is, b satisfies the definition of accumulation point.

EXERCISE 1.30 *(uniqueness inverse function)*
Assume $f : D \to R$ is invertible. Show, using the definition, that the inverse function is uniquely determined. Hint: assume f has two inverse functions. Show they have to coincide.

ANSWER 1.30
Assume that both $g(y)$ and $h(y)$ are inverse functions of f. By definition, they satisfy

$$\begin{aligned} &\text{i) } (fog)(y) = y,\ (foh)(y) = y,\ \text{for all } y \in R,\ \text{and} \\ &\text{ii) } (gof)(x) = x,\ (hof)(x) = x,\ \text{for all } x \in D. \end{aligned}$$

Using i) and ii), we obtain

$$g(y) = g(foh(y)) = (gof)(h(y)) = h(y) \text{ for all } y \in R.$$

EXERCISE 1.31 The polynomial $p(x)$ of degree 3 has zeros at $x = 1$ and $x = 2$. Furthermore, $p(3) = 4$. Determine $p(x)$. Hint: use the property that if a polynomial $p_n(x)$ of degree n has a zero at $x = a$, $p_n(x) = (x - a)p_{n-1}(x)$, where p_{n-1} is a polynomial of degree $n - 1$.

ANSWER 1.31
$p(x) = (x - 1)(x - 2)(x - a)$. As $p(3) = 4$ it follows $4 = 2 * 1 * (3 - a) \to a = 1$.

EXERCISE 1.32 *(Limits and continuity)*
The intuition that a sequence $s_n,\ n = 1, 2, 3, \cdots$ approaches the number s (or: the limit of the sequence $\{s_n\}_{n=1}^{\infty}$ is s) is formal defined as: for every $\epsilon > 0$ (however small that may be) we can find a number N such that, for all $n > N$ the distance between s_n and s, that is $|s_n - s|$, is smaller than this ϵ.
Use this definition to show that
a. $f(x) = 1$ is continuous on $\mathbb{R}$.
b. $f(x) = x$ is continuous on $\mathbb{R}$.
c. $f(x)$ is discontinuous at $x = 1$ if $f(x) = 2$ for $x \in [0, 1)$ and $f(x) = 3$ for $x \in [1, 2)$.

ANSWER 1.32
a. Let $x_0 \in \mathbb{R}$ be arbitrary, and suppose that s_n approaches x_0. Let $\epsilon > 0$ be given. By definition, there exists an N_ϵ such that for all $n > N_\epsilon$, we have $|s_n - x_0| < \epsilon$.
Now consider the distance between $f(s_n)$ and $f(x_0)$. For all $n > N_\epsilon$, we have $|f(s_n) - f(x_0)| = |1 - 1| = 0$. Hence, $f(s_n)$ approaches $f(x_0)$ whenever s_n approaches x_0.
b. Let $x_0 \in \mathbb{R}$ be an arbitrary number, and assume s_n approaches x_0. Choose an arbitrary $\epsilon > 0$. By definition, we can find an N_ϵ such that for all $n > N_\epsilon$, $|s_n - x_0| < \epsilon$. Next consider the distance between $f(s_n)$ and $f(x_0)$. Note that for all $n > N_\epsilon$, $|f(s_n) - f(x_0)| = |s_n - x_0| < \epsilon$. So, whenever

the distance between s_n and x_0 becomes small, the distance between $f(s_n)$ and $f(x_0)$ becomes small too.

c. Consider $s_n = 1 - \frac{1}{n}$. Then s_n approaches 1. But, $f(s_n) = 2$ for all n, whereas $f(1) = 3$. So, $f(s_n)$ does not approach $f(1)$. Therefore f is discontinuous at 1.

EXERCISE 1.33 *(Differentiability implies continuity)*

Formal a function f is called differentiable at x_0 if the difference quotient at this point approaches some number if the change in the function argument, Δx, approaches zero. Show that if f is differentiable at x_0, this implies that $f(x_0 + \Delta x)$ approaches $f(x_0)$ if Δx approaches zero. Or, stated differently, f is continuous at x_0.

ANSWER 1.33

By definition, f is differentiable at x_0 if

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

approaches some number s if Δx approaches zero. Equivalently,

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} - s \text{ approaches zero if } \Delta x \text{ approaches zero,}$$

or

$$\frac{f(x_0 + \Delta x) - f(x_0) - s\Delta x}{\Delta x} \text{ approaches zero if } \Delta x \text{ approaches zero.}$$

Since the denumerator tends to zero as Δx approaches zero, this fraction can only converge to zero if the numerator also tends to zero. That is,

$$f(x_0 + \Delta x) - f(x_0) - s\Delta x\text{approaches zero if } \Delta x \text{ tends to zero.}$$

This implies that

$$f(x_0 + \Delta x) - f(x_0) \text{ must approach zero if } \Delta x \text{ approaches zero,}$$

which shows that f is continuous at x_0.

EXERCISE 1.34 *(First-order condition optimum)*

Show, using the definition of differentiability (see Exercise 1.32), that if f is differentiable on an open interval I, and f attains a maximum at $x_0 \in I$, $f^{'}(x_0) = 0$. Hint, show this by contradiction. That is, assume either $f^{'}(x_0) > 0$ or $f^{'}(x_0) < 0$ and show that under those assumptions f cannot attain a maximum at x_0.

ANSWER 1.34

Assume, for example, that $f^{'}(x_0) > 0$. Since f is differentiable, the difference quotient

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

approaches the number $s := f^{'}(x_0)$ as Δx approaches zero.

Now consider approaching x_0 from the right, i.e. $\Delta x > 0$. Then the difference quotient is close to s when Δx is sufficiently small. Since $s > 0$, there must exist some $\overline{\Delta x}$ such that for all $\Delta x \in (0, \overline{\Delta x})$,

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} > 0.$$

This implies that

$$f(x_0 + \Delta x) - f(x_0) > 0 \text{ for all } \Delta x \in (0, \overline{\Delta x}),$$

or equivalently,

$$f(x) > f(x_0) \text{ for all } x \in (x_0, x_0 + \overline{\Delta x}).$$

Thus, f cannot have a maximum at x_0.

A similar argument shows that the assumption $f^{'}(x_0) < 0$ leads to a contradiction when approaching x_0 from the left.

EXERCISE 1.35 *(Rolle's theorem)*

Consider the continuous function f on the compact interval $[a, b]$. Assume $f(x)$ is differentiable on the open interval (a, b) and $f(a) = f(b)$. Below you are asked to show that there exists a point $c \in (a, b)$ for which $f^{'}(c) = 0$.

a. Motivate why f attains both a maximum M and a minimum m on $[a, b]$.

b. Assume $f(a) = M$ and $f(b) = m$. Show that there exists a point $c \in (a, b)$ for which $f^{'}(c) = 0$.

c. Assume either M or m is not attained at an endpoint of the interval. Show that there exists a point $c \in (a, b)$ for which $f^{'}(c) = 0$.

d. Assume $h(0) = h(1) = 0$ and $h(x)$ is differentiable on $(0, 1)$. Consider the function $f(x) = e^{-x^3-x^2-x}h(x)$ on $[0, 1]$. Motivate, using Rolle's Theorem, why there exists a point $c \in (0, 1)$ such that $f^{'}(c) = 0$.

ANSWER 1.35

a. Follows by Weierstrass' Theorem.

b. As $f(a) = f(b) = M = m$ it follows that $f(x)$ is constant. So, $f^{'}(x) = 0$ on (a, b).

c. Assume the maximum M is not attained at an endpoint. Then at some point $c \in (a, b)$, $f(c) = M$. As f is differentiable on (a, b), it follows that $f^{'}(c) = 0$.

d. Note: $f(0) = f(1)$. Furthermore, f is differentiable on $(0, 1)$ as both $e^{-x^3-x^2-x}$ and $h(x)$ are differentiable on $(0, 1)$. So, the conditions of Rolle's theorem are satisfied and it follows by this theorem that there exists a point $c \in (0, 1)$ such that $f^{'}(c) = 0$.

EXERCISE 1.36 *(Mean Value Theorem)*

Consider the continuous function f on the compact interval $[a, b]$. Assume $f(x)$ is differentiable on the open interval (a, b). Below, in item c., you are asked to show that there exists a point $c \in (a, b)$ for which $f^{'}(c) = \frac{f(b)-f(a)}{b-a}$.

a. Consider $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x - a)$. Show that $g(a) = g(b)$.

b. Use Rolle's Theorem (Exercise 1.34) to conclude that there exists a point $c \in (a, b)$ for which $g^{'}(c) = 0$.

c. Show, using item b., that there exists a point $c \in (a, b)$ for which $f^{'}(c) = \frac{f(b)-f(a)}{b-a}$.

d. Assume $f(x)$ is continuous on $[0, 1]$, differentiable on $(0, 1)$ and $f^{'}(x) \in [-2, 2]$. Show that $-2 + f(0) \leq f(1) \leq 2 + f(0)$.

ANSWER 1.36

a. $g(a) = f(a) - \frac{f(b)-f(a)}{b-a}(a - a) = f(a)$; $g(b) = f(b) - \frac{f(b)-f(a)}{b-a}(b - a) = f(a)$.

b. $g(a) = g(b)$ and g is differentiable on (a, b). So by Rolle's theorem there exists a $c \in (a, b)$ such that $g^{'}(c) = 0$.

c. Note $g^{'}(c) = f^{'}(c) - \frac{f(b)-f(a)}{b-a}$. The result follows then by item b.

d. f satisfies the conditions of the mean value theorem on $[0, 1] \rightarrow$ there exists a $c \in (0, 1)$ such that $f^{'}(c) = \frac{f(1)-f(0)}{1-0} = f(1) - f(0)$. So, $f(1) = f(0) + f^{'}(0)$. As $f^{'}(x) \in [-2, 2]$ it follows that $-2 + f(0) \leq f(1) \leq 2 + f(0)$.

EXERCISE 1.37 *(Monotonicity and derivative)*

Assume f is differentiable on an open interval I. Show, using the definition of differentiability (see Exercise 1.32), that f is increasing on I if and only if $f^{'}(x) \geq 0$. Hint: use the mean value theorem (Exercise 1.35) to estimate the average change in f between any two points in I.

ANSWER 1.37

" $\Leftarrow$ " Let a and b be any two points in I with $a < b$. Consider $\frac{f(b)-f(a)}{b-a}$.

Note that f is continuous on $[a, b]$ and differentiable on (a, b). By the Mean Value Theorem, there exists a $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f^{'}(c).$$

By assumption, $f^{'}(c) \geq 0$. Since $b - a > 0$, it follows that $f(b) - f(a) \geq 0$, i.e., $f(b) \geq f(a)$.

" $\Rightarrow$ " Let $x_0 \in I$ and take $b > x_0$. Since f is monotonically increasing, $f(b) \geq f(x_0)$, and hence $\frac{f(b)-f(x_0)}{b-x_0} \geq 0$.

Now consider $b < x_0$. Then $f(b) \leq f(x_0)$, so again $\frac{f(b)-f(x_0)}{b-x_0} \geq 0$.

Thus,

$$\text{for every } \Delta x \neq 0, \ \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} \geq 0.$$

Since f is differentiable on I, it follows that when Δx approaches zero, $\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ approaches $f^{'}(x_0)$. Since for every Δx the difference quotient is nonnegative, we conclude that $f^{'}(x_0)$ must be nonnegative too.

EXERCISE 1.38 *(First-order characterization convex function)*

Assume f is differentiable on an open interval I. Below, in items c.and e., respectively, you are asked to show that f is convex on I if and only if $f(y) \geq f(x) + f^{'}(x)(y-x)$ for all $x, y \in I$.

a. Consider x_0 and $y = x_0 + \Delta x \in I$. Use the definition of convexity to conclude that $(1-\lambda)f(x_0+\Delta x) \geq f(x_0 + (1-\lambda)\Delta x_0) - f(x_0) + (1-\lambda)f(x_0)$, for each $\lambda \in [0,1]$.

b. Let $t := 1-\lambda$. Show that we obtain the next inequality from item a.: $f(x_0+\Delta x) \geq f(x_0) + \frac{f(x_0+t\Delta x_0)-f(x_0)}{t\Delta x_0}\Delta x_0$.

c. Use the definition of differentiability (see Exercise 1.32) to show from item b., that if f is convex, $f(y) \geq f(x_0) + f^{'}(x_0)(y-x_0)$.

d. Assume $f(y) \geq f(x) + f^{'}(x)(y-x)$ for all $x, y \in I$. Let $z := \lambda x + (1-\lambda y)$. Show that (i) $f(x) \geq f(z) + f^{'}(z)(x-z)$ and (ii) $f(y) \geq f(z) + f^{'}(z)(y-z)$.

e. By multiplication of (i) by λ and (ii) by $(1-\lambda)$ in item d., and adding both resulting inequalities show that if $f(y) \geq f(x) + f^{'}(x)(y-x)$ for all $x, y \in I$, f is convex on I.

ANSWER 1.38

a. By the definition of convexity

$$f(\lambda x_0 + (1-\lambda)y) \leq \lambda f(x_0) + (1-\lambda)f(y).$$

If we take $y = x_0 + \Delta x$, this becomes

$$f(x_0 + (1-\lambda)\Delta x) \leq \lambda f(x_0) + (1-\lambda)f(x_0+\Delta x_0) + f(x_0) - f(x_0).$$

Rearranging, we get

$$(1-\lambda)f(x_0+\Delta x) \geq f(x_0 + (1-\lambda)\Delta x_0) - f(x_0) + (1-\lambda)f(x_0).$$

b. Setting $t = (1-\lambda)$, the inequality from item a. becomes:

$$tf(x_0+\Delta x) \geq f(x_0+t\Delta x_0) - f(x_0) + tf(x_0),$$

which can be rewritten as

$$f(x_0+\Delta x) \geq \frac{f(x_0+t\Delta x_0) - f(x_0)}{t} + f(x_0) = \frac{f(x_0+t\Delta x_0) - f(x_0)}{t\Delta x_0}\Delta x_0 + f(x_0).$$

c. Since f is differentiable on I,

$$\frac{f(x_0 + t\Delta x_0) - f(x_0)}{t\Delta x_0} \text{ approaches } f^{'}(x_0) \text{ if } t \text{ approaches zero.}$$

Because the inequality in item b. holds for all $t \in [0, 1]$, we can in particular consider t approaching zero to obtain

$$f(x_0 + \Delta x) \geq f^{'}(x_0)\Delta x_0 + f(x_0)$$

or, equivalently, using $y = x_0 + \Delta x$, $f(y) \geq f(x_0) + f^{'}(x_0)(y - x_0)$.

d. For $x, y \in I$, also $z := \lambda x + (1 - \lambda y) \in I$. Using the definition of convexity with respect to the pairs (x, z) and (y, z), respectively, we obtain both inequalities.

e. From (i):

$$\lambda f(x) \geq \lambda f(z) + \lambda f^{'}(z)(x - z),$$

and from (ii):

$$(1 - \lambda)f(y) \geq (1 - \lambda)f(z) + (1 - \lambda)f^{'}(z)(y - z).$$

Adding these two inequalities gives:

$$\lambda f(x)+(1-\lambda)f(y) \geq \lambda f(z)+\lambda f^{'}(z)(x-z)+(1-\lambda)f(z)+(1-\lambda)f^{'}(z)(y-z) = f(z) = f(\lambda x+(1-\lambda)y).$$

EXERCISE 1.39 *(Taylor's theorem)*

Assume $f(x)$ is n times differentiable on the open interval I and $a \in I$. Below, in item d., you are asked to show that for any $x \in I$ there exists a point ξ located between x and a such that

$$f(x) = p_{n-1}(x) + \frac{f^{(n)}(\xi)}{n!}(x - a)^n, \tag{1.1}$$

where $p_{n-1}(x)$ is given by next polynomial of degree $n - 1$

$$p_{n-1}(x) := f(a) + \frac{f^{'}(a)}{1!}(x - a) + \frac{f^{"}(a)}{2!}(x - a)^2 + \frac{f^{(3)}(a)}{3!}(x - a)^3 + \cdots + \frac{f^{(n-1)}(a)}{(n - 1)!}(x - a)^{(n-1)}.$$

a. Let $F(y) := f(x) - \left(f(y) + \frac{f^{'}(y)}{1!}(x - y) + \frac{f^{"}(y)}{2!}(x - y)^2 + \cdots + \frac{f^{(n-1)}(y)}{(n-1)!}(x - y)^{(n-1)}\right)$.
Consider $g(y) := F(y) - \left(\frac{x-y}{x-a}\right)^n F(a)$. Show that $g(a) = g(x) = 0$.

b. Use Rolle's Theorem (Exercise 1.34) to conclude that there exists a point ξ between x and a such that $F^{'}(\xi) + \frac{n(x-\xi)^{n-1}}{(x-a)^n}F(a) = 0$.

c. Show that $F^{'}(y) = -\frac{(x-y)^{n-1}}{(n-1)!}f^{(n)}(y)$.

d. Calculate $F(a)$ from item b. Next show that there exists a point ξ between x and a such that (1.1) holds.