POLYHEDROIDS: THAT FAMILY OF POLYHEDRA

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To my wife Bep
without whose enduring encouragement
this book may never have come to an end

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PREFACE

Polyhedroids

In the context of this book, polyhedra are defined as portions of space that entirely are surrounded by regular polygons. The main emphasis is placed on the so-called uniform polyhedra.
Polyhedra are called uniform if the faces meet in the same manner at every vertex: is face transitive [0.5]. Under this definition fall: the 5 Platonic solids, the 13 Archimedean solids (see Chapter 1), the 4 regular stellated polyhedra, called Kepler-Poinsot polyhedra, and 53 uniform star polyhedra [1.5] (Chapter 7). Besides these there are two infinite rows of uniform prisms and antiprisms (Chapter 6). Only a limited group is convex: the Platonic and Archimedean solids and the prisms and antiprisms.
Many other configurations are derived from these polyhedra or have very much in common. It is therefore important to have a proper knowledge of their basic geometry. In this book attention will be paid to the Platonic and Archimedean solids themselves, but also to a number of polyhedron based structures, in particular domes (Chapter 8) and space frames. In general we have called this group of forms 'Polyhedroids'. They together form a great family, hence the title of this book.

Fig. 0.1. *Many members of the 'Family of Polyhedra'*

In fact this study started with the wish to try to understand, what all these figures have in common and how they can be materialized in numbers and in visual form. For the polyhedra, composed of regular polygons, the author found a basic approach in the book *Vielecke und Vielfläche* by M. Brückner [1.2]. From this starting point many other groups could be derived, such as their reciprocals or duals, prisms and antiprisms, stellated polyhedra and finally sphere subdivisions. A proper knowledge of their geometry learns how these can be manipulated or be combined in spatial configurations: packings and spatial structures (Chapter 13).

Most of the pictures in this book have been made with the computer programme CORDIN, developed by the author in close co-operation with Gerrit van der Ende. The first versions of this programme date from as long ago as 1975 and it was originally only meant to have the availability of a means, with which polyhedral figures could be calculated, either resulting in numeric data or in pictures. But it gradually evolved into a quite versatile instrument and in many previous occasions, in conference papers and in articles, examples were shown proving its potential. The projection of vector files on the surface of polyhedra is a recent new extension, although it is still a rather laborious process in its present form. The author therefore decided to provide so many data that most of the processes described can be traced following the course that the author himself went, he not being a mathematician but - as he preferably calls it - a simple structural designer with an architectural background. Its understanding therefore does not require a thorough knowledge of mathematics but only the availability of a sufficient amount of common sense - and patience of course. He describes this from his personal background and experiences and it does not claim to cover the total area. Much additional information can be found on Internet or from other sources.

In this book thus not only the geometrical aspects are discussed but also great emphasis is given to illustrations. In Chapter 16, a number of stereo pictures is provided that can be seen as so-called anaglyphs (with additionally available blue/red-glasses). When printed, the anaglyphs loose some in quality, as it is difficult to print the colours exactly as needed. Printing in colour may be considered, but as such this is costly and therefore we choose in the first instance to publish it in the form of an electronic book to be read on I-pads or tablets. This opened the possibility to include hyperlinks in the text that allow direct access to Internet. In printed form these links do not work this way of course, but they are still worth looking at.

Some of the exercizes described, may at first glance seem playful and not very serious, originating from the mind of the author, but he believes that such an attitude is required to come to new ideas and to new experiments. As examples of this thesis may serve the development of spatial structures as described in Chapter 13, that almost by definition are based on polyhedral geometry, as well as the new soccer ball design that is shown in Chapter 12, that deals with isodistant polyhedra.

References

[0.1] http://en.wikipedia.org/wiki/Uniform_polyhedron

[0.2] Huybers, P., 'The polyhedral world', Chapter in: 'Beyond the Cube: The Architecture of Space frames and Polyhedra', by J.F. Gabriel editor, John Wiley and Sons, Inc., New York, 1997, p. 243-279.

[0.3] Daniela Bertol, 'Polyhedra' and 'Polyhedra 2', available as iBooks at:

[0.4] https://itunes.apple.com/us/book/polyhedra/id554504060?mt=11

Chapter 1. THE CONVEX UNIFORM POLYHEDRA

1. 1. The geometry of polyhedra

First of all we must agree upon a workable definition of what we consider in this context as a convex uniform polyhedron [Ref. 1.5]. We assume that:

1) They are covered with a closed pattern of plane, regular polygons.
   At this point we shall look only at the so-called Platonic solids, that are composed of identical polygons, and at the Archimedean solids, that consist of two or three different kinds of polygons. Both groups are called after the ancient scientists to which their discovery is usually ascribed [Refs. 1.1, 1.3, 1.6, 1.9]. The different polygons that occur in these solids have either 3, 4, 5, 6, 8 or 10 edges. The endless rows the prisms and antiprisms also fall in this category, but their two parallel sides can have any number of edges, and they will be treated separately in Chapter 6.
2) All vertices of a polyhedron lie on one circumscribed sphere.
3) These vertices all are identical. This is so because around each vertex of a particular polyhedron the polygons are grouped in the same number, kind and order of sequence.
4) The polygons meet in pairs at a common edge.
5) The dihedral angle at such an edge is always convex. This means that the dihedral angle between two adjacent polygons is less than 180°, if seen from the interior, or in other words: the sum of the polygon face angles that meet at a vertex is always smaller than 360° (see Table 1.6).

1.2. The different kinds

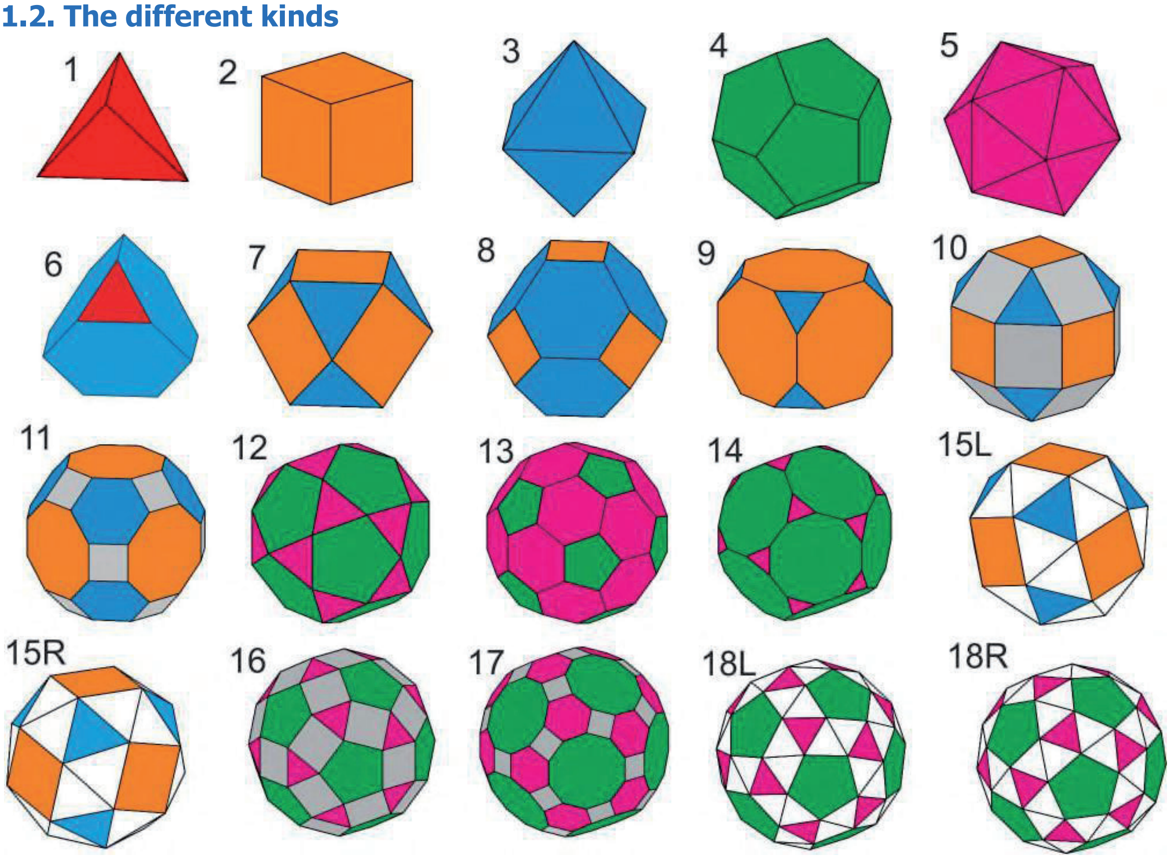


Fig. 1.1. *Review of the 5 regular polyhedra and the 13 semi-regular polyhedra, of which two have a left-handed as well as a right-handed version (P15 and P18).*

These are:
The 5 regular polyhedra: 1) Tetrahedron, 2) Cube, 3) Octahedron, 4) Dodecahedron, 5) Icosahedron
The 13 semi-regular polyhedra: 6) Truncated Tetrahedron, 7) Cuboctahedron, 8) Truncated Octahedron, 9) Truncated Cube, 10) Rhombicuboctahedron,11) Truncated Cuboctahedron,
12) Icosidodecahedron, 13) Truncated Icosahedron, 14) Truncated Dodecahedron, 15) Left and Right handed Snub Cube, 16) Rhombi-cosidodecahedron, 17) Truncated Icosidodecahedron, 18) Left and Right handed Snub Dodecahedron.

It is easy to understand that under these conditions the minimum total number of polygons around a vertex is 3, the maximum number 5 and it is also simple to prove, that not more than 5 totally regular polyhedra can exist (Fig. 15.7). These are the regular or Platonic solids and they are each composed of one kind of face. Polyhedra are called semi-regular, or Archimedean, if more than one kind of polygon is used for their construction. According the first condition of the previous definition - namely, that the polygon has no more than 3, 4, 5, 6, 8 or 10 edges - a group of 15 principally different semi-regular polyhedra is found. For more information see the links: http://en.wikipedia.org/wiki/Platonic_solid and http://en.wikipedia.org/wiki/Archimedean_solid

P	Vertex code	Name	Numbers of polygons by side number						Total Faces, Edges, Vertices			Radius
			3	4	5	6	8	10	F	E	V	
1	3-3-3	Tetrahedron	4	-	-	-	-	-	4	6	4	0.61237244
2	4-4-4	Cube	-	6	-	-	-	-	6	12	8	0.86602540
3	3-3-3-3	Octahedron	8	-	-	-	-	-	8	12	6	0.70710678
4	5-5-5	Dodecahedron	-	-	12	-	-	-	12	30	20	1.40125854
5	3-3-3-3-3	Icosahedron	20	-	-	-	-	-	20	30	12	0.95105652
6	3-6-6	Truncated Tetrahedron	4	-	-	4	-	-	8	18	12	1.17260394
7	3-4-3-4	Cuboctahedron	8	6	-	-	-	-	14	24	12	1.00000000
8	4-6-6	Truncated Octahedron	-	6	-	8	-	-	14	36	24	1.58113883
9	3-8-8	Truncated Cube	8	-	-	-	6	-	14	36	24	1.77882365
10	3-4-4-4	Rhombicuboctahedron	8	18	-	-	-	-	26	48	24	1.39896633
11	4-6-8	Truncated Cuboctahedron	-	12	-	8	6	-	26	72	48	1.31761091
12	3-5-3-5	Icosidodecahedron	20	-	12	-	-	-	32	60	30	1.61803399
13	5-6-6	Truncated Icosahedron	-	-	12	20	-	-	32	90	60	1.47801866
14	3-10-10	Truncated Dodecahedron	20	-	-	-	-	12	32	90	60	1.96944902
15	3-3-3-3-4	Snub Cube	32	6	-	-	-	-	38	60	24	1.34371337
16	3-4-5-4	Rhombicosidodecahedron	20	30	12	-	-	-	62	120	60	1.23295051
17	4-6-10	Truncated Icosidodecahedron	-	30	-	20	-	12	62	180	120	3.80239450
18	3-3-3-3-5	Snub Dodecahedron	80	-	12	-	-	-	92	150	60	1.15583738

Table 1.1*. Some characteristic aspects of the Platonic and Archimedean polyhedra.*

In the table a few characteristics of polyhedra are given, where table P = polyhedron index, Vertex code = side-numbers of respective polygons that meet in a vertex; V, E and F = number of vertices, edges and faces. Radius = radius of circumscribed sphere at unit edge length. The formula of Euler is applicable, which means that: V - E + F = 2.

The different polyhedra are further referred to as P#, with # for the index number. These index numbers are useful to indicate the individual polyhedra, in order to avoid the need to use their mostly difficult and sometimes long scientific names. In computer programmes for the calculation of their geometry or for their visual presentation it is necessary to indicate them by a unique number.

That is why they are numbered here in a certain order of sequence, that is dictated by the following consecutive criteria:

[1] Number of faces
[2] Number of edges
[3] Radius of the circumscribed sphere

If only criteria 1 and 2 were applied, the truncated dodecahedron and the truncated icosahedron would have obtained the same number. The left- and right-handed snubs have the same identification numbers, because they are topologically identical, although they have different co-ordinates. They are sometimes called 'chiral' (see Chapter 4) .

1.3. The Platonic solids

There is a direct geometric relation between the regular polyhedra. P1, P3, P4 and P5 are for instance inscribable in Cube P2, following Figs. 1.2 and 1.3. This picture also gives the clues for the definition of their positions in the Euclidean space. The mutual relations are expressed in numerical form or in the form of simple expressions in Table 1.2.

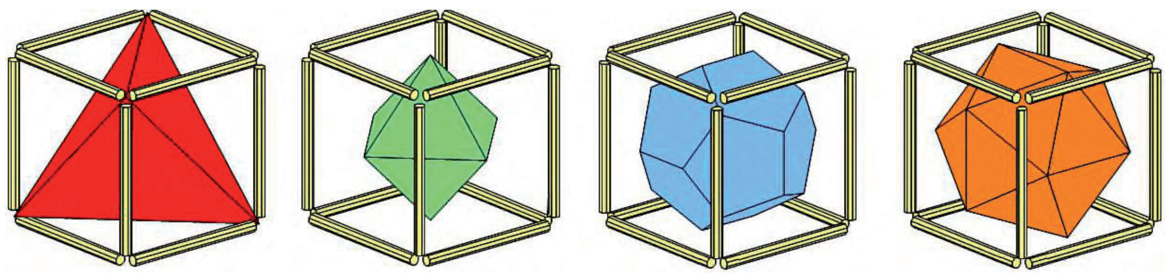


Fig. 1.2. *The relations of the Platonic polyhedra.*

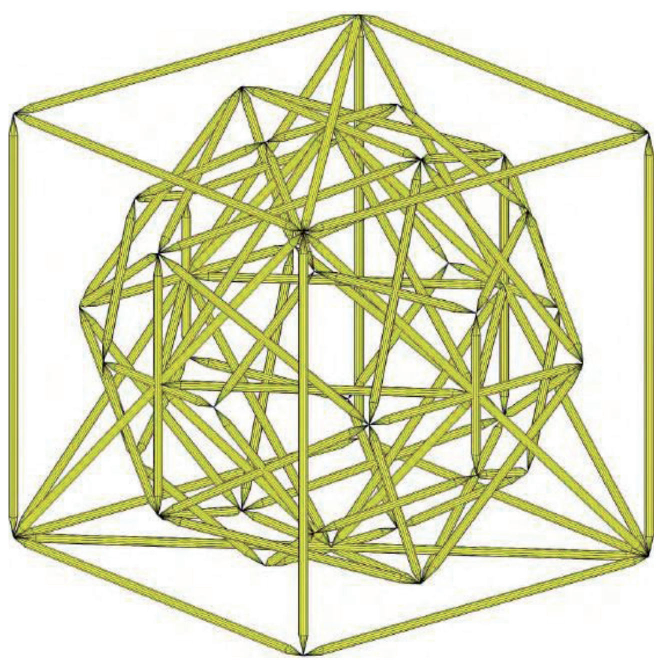
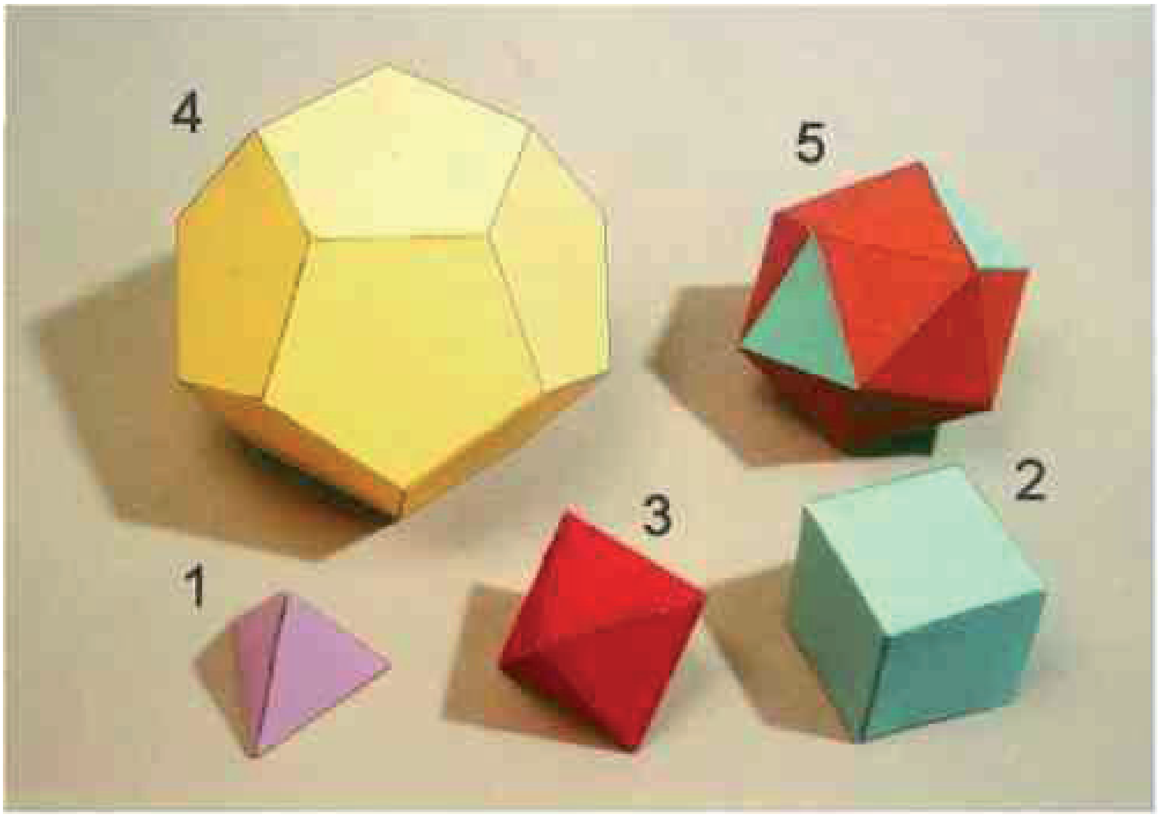


Fig. 1.3. *The inscribability of the other four solids in the cube*



Fig, 1.4. *Models of the Platonic polyhedra*

In this table the value of τ, also known as the Golden Section, is defined as:
$\tau = (1 + \sqrt{5}) / 2 = 1.6180339887499....$
This has a few odd characteristics, a.o.
$\tau^2 = \tau + 1 = 2.61803399$
$1/\tau = \tau - 1 = 0.61803399$

cube	tetrahedron	octahedron	dodecahedron	icosahedron
1 1.00000000	$\sqrt{2}$ 1.41421356	$1 : \sqrt{2}$ 0.70710678	$1 : (\tau + 1) = 1 : \tau^2$ 0.381996601	$1 : \tau = \tau - 1$ 0.61803399

Table 1.2. *The relations of the 5 regular polyhedra.*

1.4. The Archimedean solids

The names of the semi-regular solids indicate that they are generally considered as to be derived from the regular solids by truncation. If this truncation is done so that the original face edges are divided in three parts, the original faces convert into polygons with double the number of sides (i.e. triangle becomes hexagon, square becomes octagon and pentagon becomes decagon). Thus five new polyhedra are found: the truncated versions of the regular solids (P6, P8, P9, P13 and P14).

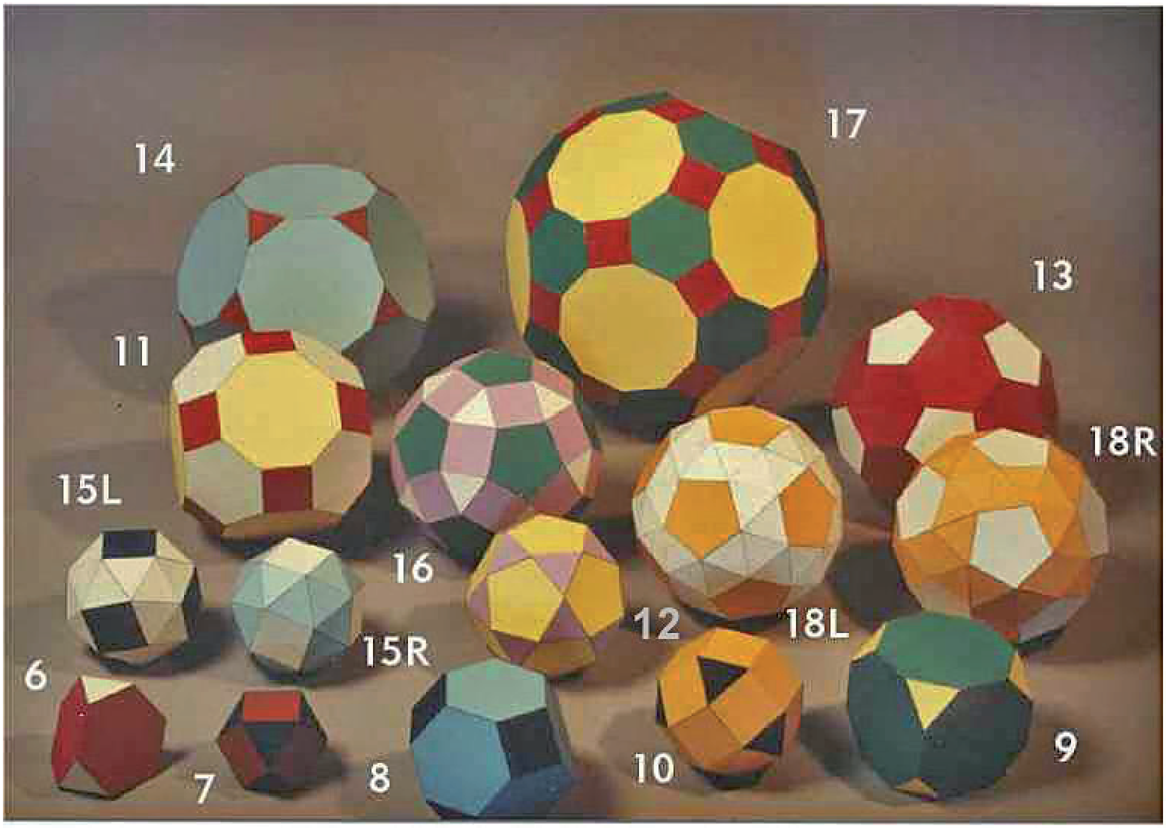


Fig. 1.5. *The 15 Archimedean polyhedra. P15 and P18 have a left-handed and a right-handed version.*

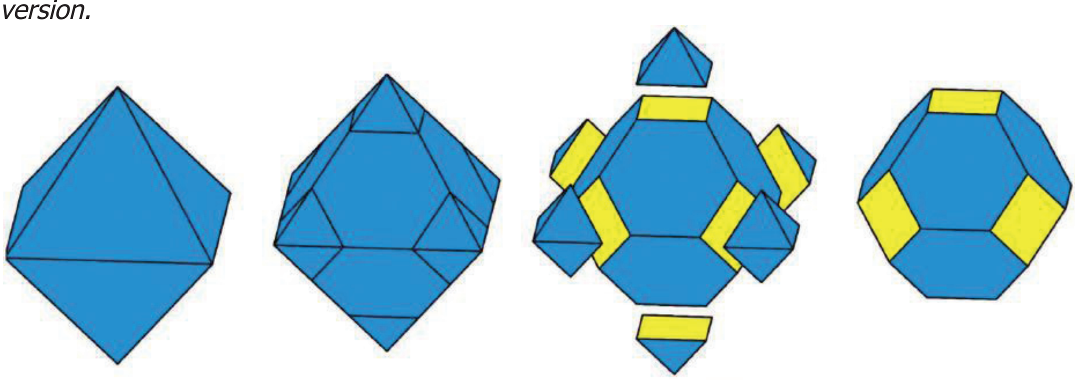


Fig. 1.6. *The truncation of the octahedron P3 at one third of its side length, forming P8*

The truncation procedure can be carried out a little bit further so that the original edges are exactly bisected. This gives two new solids (see Fig. 1.5): the Cuboctahedron (P7) and the Icosidodecahedron (P12). These two are peculiar ones and they are called quasi-regular, solids because they can be considered, as their names already suggest, to be compounds of two pairs of regular solids. P7, the Cuboctahedron, is composed of 6 squares (like the cube P2) and 8 triangles (like the octahedron P3). P12, the Icosidodecahedron, is composed of 20 triangles (like the Icosahedron P5) and 12 pentagons (like the Dodecahedron P4).

Truncation can also take place parallel to the edges. This generally produces square extra faces and it yields four new semi-regular solids (Nos. 10, 11, 16 and 17)

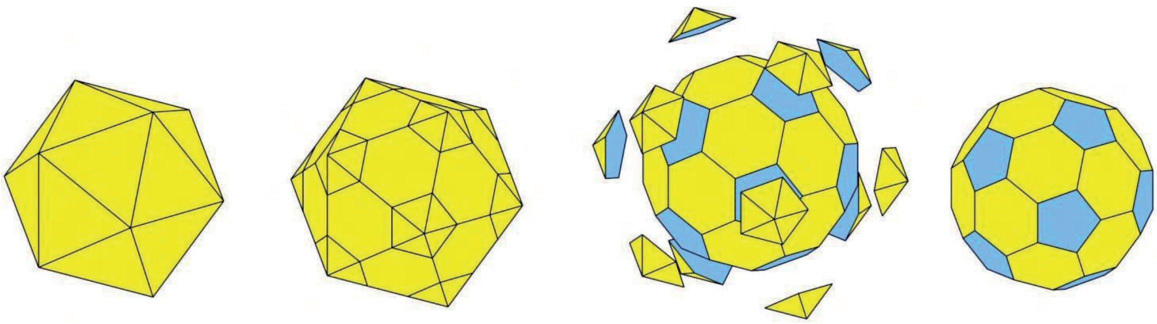
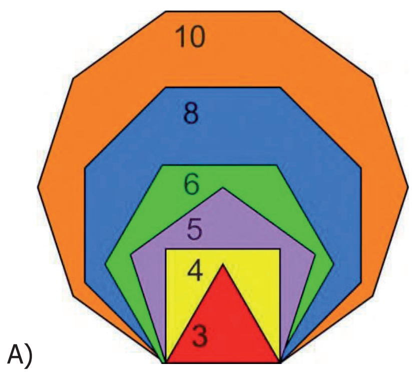


Fig. 1.7. *The formation of the truncated icosahedron at one third of the sides*

There are two other solids that are found by truncation of the corners and a double truncation of the edges. There are in fact four of them, as they occur in a right-handed as well in a left-handed (enantiomorphic) version. These are the Snub Cube (No. 15) and the Snub Dodecahedron (No. 18). These two are called after their circumscribed figures. The Snub Cube has 6 squares, each one completely surrounded by triangles, whereas the Snub Dodecahedron has 12 pentagons in a corresponding situation.

1.5. Regular polygons



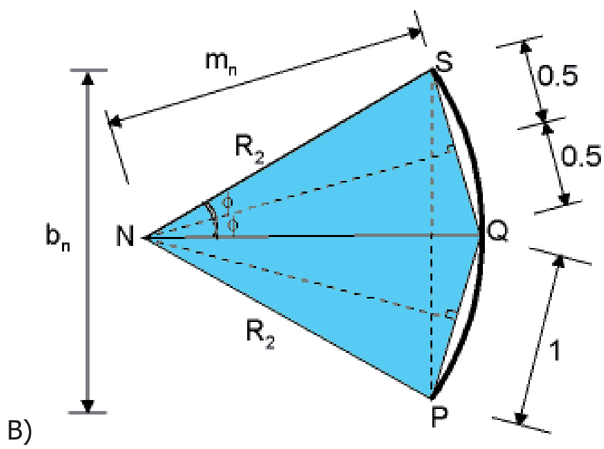


Fig. 1.8. *A) The six different polygons in polyhedra.* *B) Two adjacent sectors in a polygon.*

The central angle of a regular polygon with n sides: $\varphi_n = \dfrac{\pi}{n}$ {1.1}

The radius of the circum-circle: $R_2 = \dfrac{1}{2\sin\varphi_n}$ {1.2}

The distance of the center to the mid-point of a side: $m_n = \dfrac{1}{2\tan\varphi_n} = \sqrt{(R_2{}^2 - 0.25)}$ {1.3}

Two alternate corners of a polygon (P and S in Fig. 1.8B) have the distance:

$$b_n = 2\cos\varphi_n \qquad \{1.4\}$$

The area of an n-gon

$$A_n = \frac{1}{2} n m_n = \frac{1}{2}\sqrt{(R_2{}^2 - 0.25)} = \frac{n}{4\tan\varphi_n} \qquad \{1.5\}$$

1.6. Vertex situations

If one connects the other ends of the edges, meeting in a vertex of a polyhedron a so-called 'vertex figure' is found. It forms the basis of a pyramid with the original vertex as its apex. This cap is called 'vertex pyramid' [Ref. 1.2].

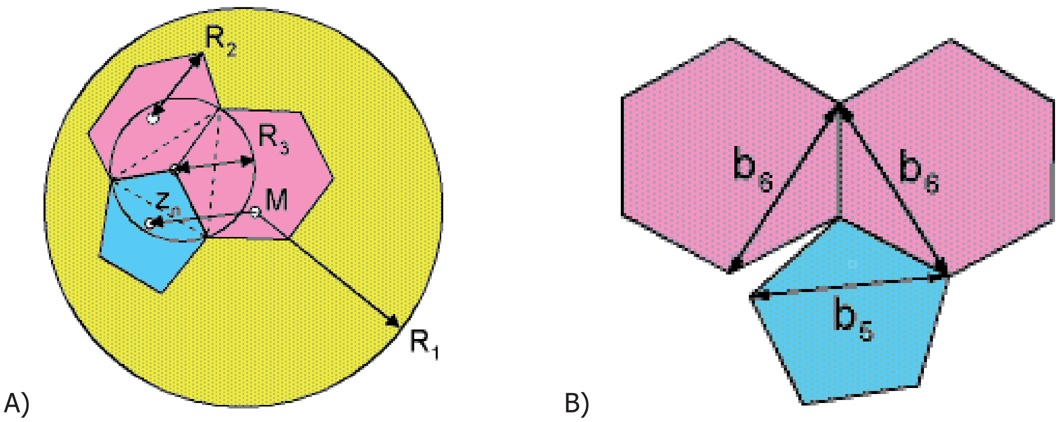


Fig. 1.9. *A) Polyhedron P13 (code 5-6-6), showing the situation in a vertex.*
B) The three polygons in P13 with their 'small diagonals' b.

n	Phi	R_2	m_n	b_n	$Area_{total}$
3	60.00000000	0.57735027	0.28867513	1.00000000	0.43301270
4	45.00000000	0.70710678	0.50000000	1.41421356	1.00000000
5	36.00000000	0.85065081	0.68819096	1.61803399	1.72047740
6	30.00000000	1.00000000	0.86602540	1.73205081	1.59807621
7	25.71428571	1.15238244	1.03826070	1.80193774	3.63391244
8	21.50000000	1.30656296	1.20710678	1.84775907	4.82842712
9	20.00000000	1.46190220	1.37373871	1.87938524	6.18182419
10	18.00000000	1.61803399	1.53884177	1.90211303	7.69420884
11	16.36363636	1.77473277	1.70284362	1.91898595	9.36563991
12	15.00000000	1.93185165	1.86602540	1.93185165	11.19615242
13	13.84615385	2.08929073	2.02857974	1.94188363	13.18576833
14	11.85714286	2.24697960	2.19064313	1.94985582	15.33450194
15	11.00000000	2.40486717	2.35231505	1.95629520	17.64236291
16	11.25000000	2.56291545	2.51366975	1.96157056	20.10935797
17	10.58823529	2.72109558	2.67476375	1.96594620	21.73549190
18	10.00000000	2.87938524	2.83564091	1.96961551	25.52076819
19	9.47368421	3.03776691	2.99633573	1.97272261	28.46518943
20	9.00000000	3.196222661	3.15687576	1.97537668	31.56875757

Table 1.3. *Relevant data of the first series of regular polygons with 3 to 20 sides.*

A vertex figure has as many edges as the number of polygons that meet in the vertex of a polyhedron. It has therefore either 3, 4 or 5 edges and its form may be regular or not.

1.6.1. The triangular vertex figures

The triangular figure has in its most general form the three different sides u, v and w. The values of these are equal to b_n in the respective adjacent polygon.

The vertex figure can thus be scalene (P11 and P17), isosceles (P6, P8, P9, P13, P14 and the prisms) or equilateral (P1, P2 and P4).

$$R_3 = \frac{uvw}{4\sqrt{s(s-u)(s-v)(s-w)}} \Rightarrow \text{with } s = \frac{1}{2}(u+v+w) \qquad \{1.6\}$$

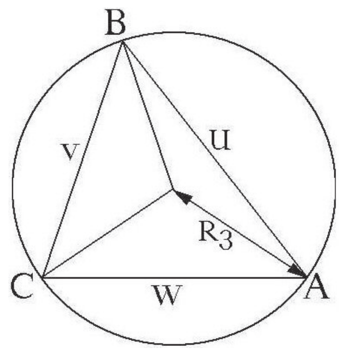


Fig. 1.10. *Principal vertex figure with 3 sides*

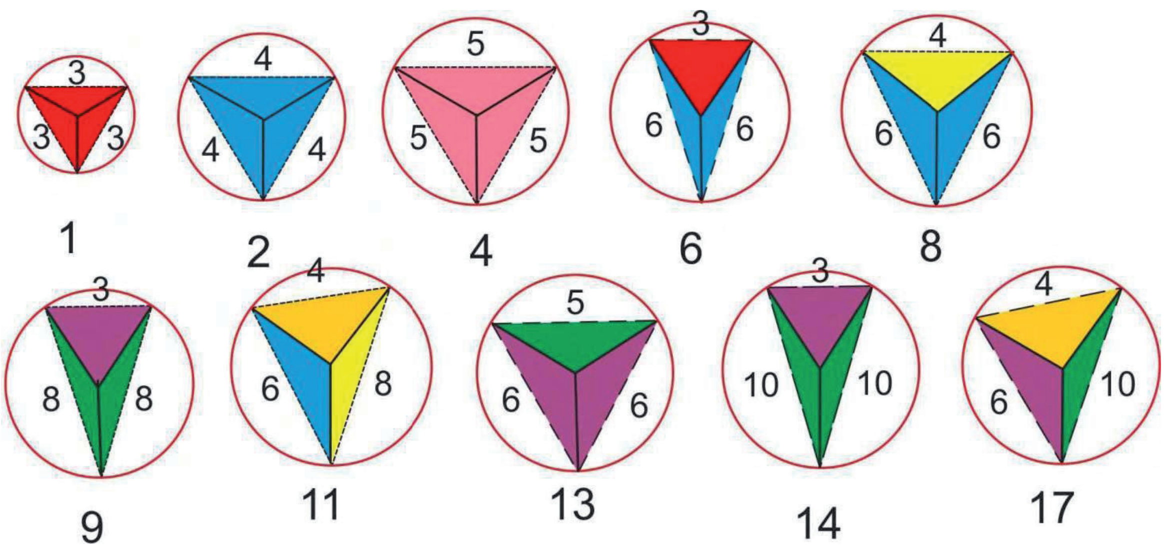


Fig. 1.11. *All triangular vertex figures*

1.6.2. The quadrangular vertex figures

The quadrangular vertex figure principally has the shape of a trapezoid (P10 and P16 and the antiprisms). R_3 in this case is again determined by the fact that the circle has to pass through the three corners A, B and C. Subsequently, the vertex figure can be reduced to a basic triangle with the sides u, v and d_m with

$$d_m = \sqrt{(v^2 + uw)} \qquad \{1.7\}$$

and similarly: $R_3 = \frac{uvd_m}{4\sqrt{t(t-u)(t-v)(t-d_m)}}$ ==> with $t = ½(u + v + d_m)$ {1.8}

For antiprisms a general expression for d_m can be derived:

$$d_m = \sqrt{(1 + 2\cos\varphi_n)} \quad \{1.9\}$$

where n is the number of the sides of the two parallel variable polygons [1.4].

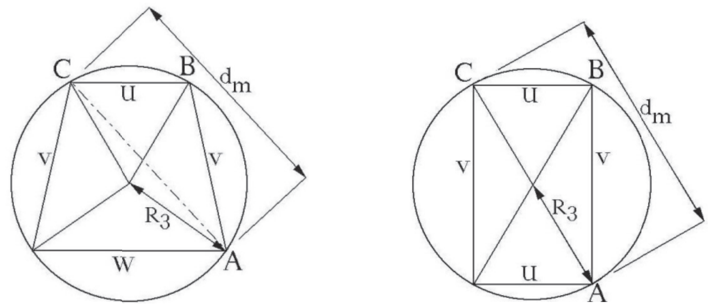


Fig. 1.11. *Principal vertex figures with 4 sides.*

Two polyhedra can be found where u = w (the so-called 'quasi-regular' solids P7 and P12) and where the vertex figure is a rectangle with the diagonal:

$$d_m = \sqrt{(u^2 + v^2)} = 2R_3$$

For the octahedron (P3): u = v = w = 1, so that the vertex figure is a square with:
$d_m = 2\ R_3 = \sqrt{2}$

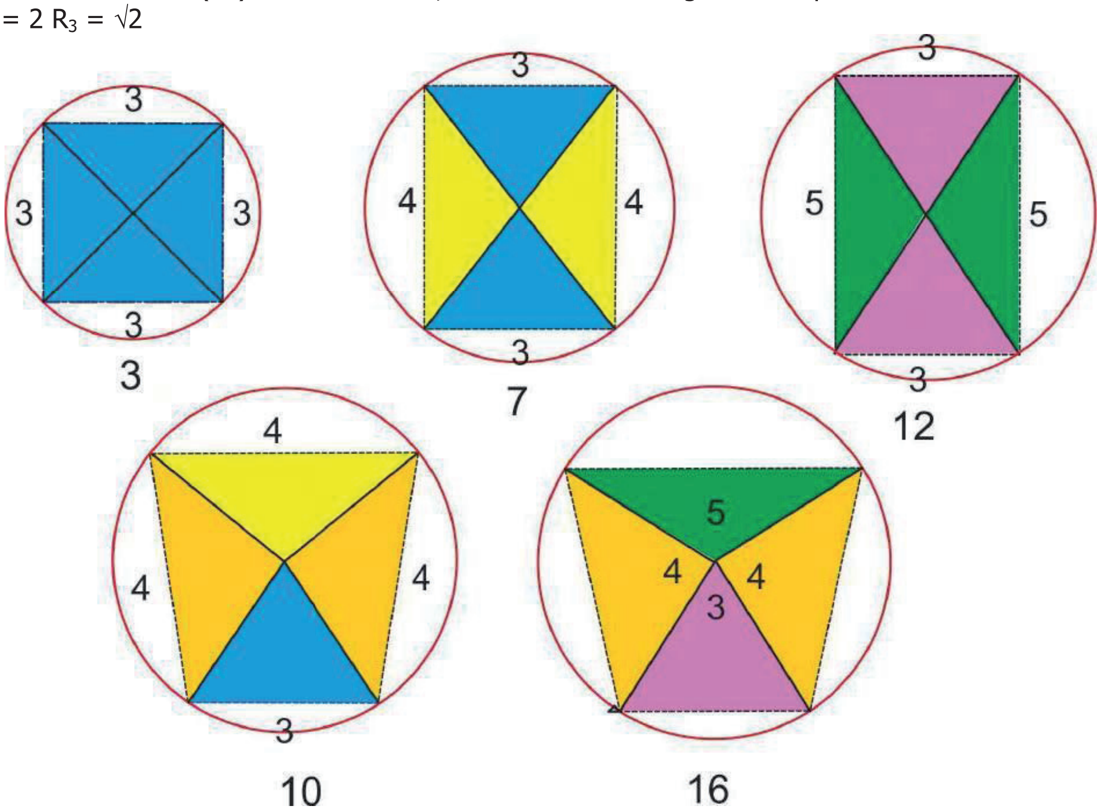


Fig. 1.13. *All quadrangular vertex figures*

1.6.3. The pentagonal vertex figures

The vertex figures of the Icosahedron P5 and of the two snub solids, P15 and P18, are pentagonal. In all three cases 4 equilateral triangles meet, the fifth meeting polygon being either a triangle, a square or a pentagon. Here again radius R_3 of the circumscribed circle is found from the basic triangle A-B-C, in which AB = BC = 1. The length of AC = d_m has to be derived with the help of a third power equation.

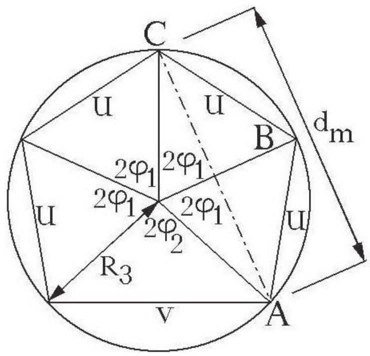


Fig. 1.14. *Principal vertex figure with 5 sides.*

$4\varphi_1 + \varphi_2 = 180^\circ$ ==> $\varphi_2 = 180^\circ - 4\varphi_1$

$\sin\varphi_1 = \dfrac{1}{2R_3}$ and $\sin\varphi_2 = \dfrac{b}{2R_3} = \dfrac{\cos\varphi_n}{R_3}$

$$R_3 = \frac{\cos\varphi_n}{\sin\varphi_2} = \frac{1}{2\sin\varphi_1} \qquad \{1.10\}$$

$2\sin\varphi_1\cos\varphi_2 = \sin\varphi_2 = \sin(4\varphi_1) = 2\sin(2\varphi_1)\cos(2\varphi_1)$, so that:

$$4\cos^3\varphi_1 - 2\cos\varphi_1 - \cos\varphi_n = 0 \qquad \{1.11\}$$

This equation can be solved for various values of n. This is done in Table 1.4 [see Ref. 1.2].

P	$\cos\varphi_1$	φ_1	$\varphi_2 = 180^\circ - 4\varphi_1$	R_3
P5	0.8091699	36	36	0.85065080
P15	0.8425092	31.59396276	49.62414896	0.92819138
P18	0.8577807	30.93168860	56.27324558	0.97273285

Table 1.4. *Values of φ_n and φ_1 in the basic pentagonal vertex figure*.

A pentagonal vertex figure can be reduced to a basic quadrangle u-u-v-d_m by the introduction of a diagonal with the value:

$$d_m = 2\cos\varphi_1$$

There are ten polyhedra with a triangular vertex figure: P1, P2, P4, P6, P8, P9, P11, P13, P14 and P17. All other eight vertex figures can be reduced also to the form of a triangle. Two sides have the length of a short diagonal in either a triangle, a square or a pentagon and the third side has a length of the form d_m, as derived in the foregoing. They are summarized in the following table 1.5.

P	Code	diagonal d_m		
P3	3-3-3-3	3-3-d_1	$\sqrt{2}$	= 1.41421356
P7	3-4-3-4	3-4-d_2	$\sqrt{3}$	= 1.73205081
P12	3-5-3-5	3-5-d_3	$\sqrt{(1+4\cos^2 36^o)}$	= 1.90211303
P10	3-4-4-4	3-4-d_4	$\sqrt{(2+\sqrt{2})}$	= 1.84775907
P16	3-4-5-4	3-4-d_5	$\sqrt{(2+2\cos 36^o)}$	= 1.90211303
P5	3-3-3-3-3	3-3-d_6	$2\cos(36^o)$	= 1.61803399
P15	3-3-3-3-4	3-3-d_7	$2\cos(31.59396280^o)$	= 1.68501832
P18	3-3-3-3-5	3-3-d_8	$2\cos(30.93168860^o)$	= 1.71556150

Table 1.5. *Values of basic diagonals*

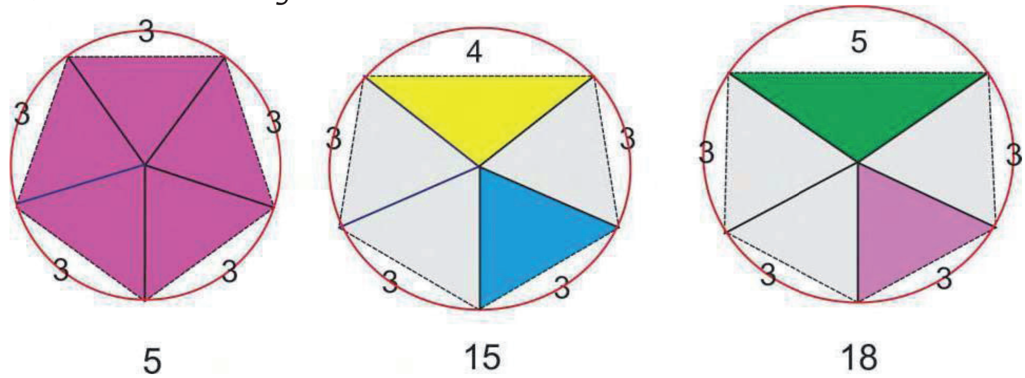


Fig. 1.15. *All pentagonal vertex figures*

For a complete review of all uniform polyhedra by vertex figure see
http://en.wikipedia.org/wiki/List_of_uniform_polyhedra_by_vertex_figure

1.7. Characteristic radii of a polyhedron

Apart from the radius of the vertex figure's circum-circle R_3, a polyhedron has a few other characteristic radii. The first is that of the circumscribed sphere of a polyhedron, which has to pass through the circle with the radius R_3 and its respective vertex T. This is shown in detail in Fig. 1.17A. R_1 is the circle around the triangle with the sides 1, 1 and 2 R_1.
Half the sum of the sides: s = R_3 + 1.

$$R_1 = \frac{2R_3}{4\sqrt{(R_3+1)(-R_3+1)R_3R_3}} = \frac{1}{2\sqrt{(1-R_3{}^2)}} \quad \text{(see table 1.16)} \qquad \{1.13\}$$

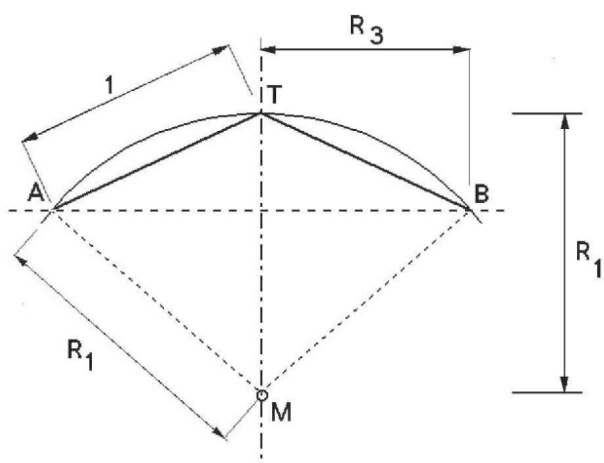


Fig.1.16. *Derivation of R_1 as a circle around the triangle A-B-T.*

The radius of the 'inter-sphere': $R_5 = \sqrt{(R_1^{\,2} - 0.25)}$ {1.14}

The distance of the n-gon from the center M: $z_n = \sqrt{(R_1^{\,2} - R_2^{\,2})}$ {1.15}

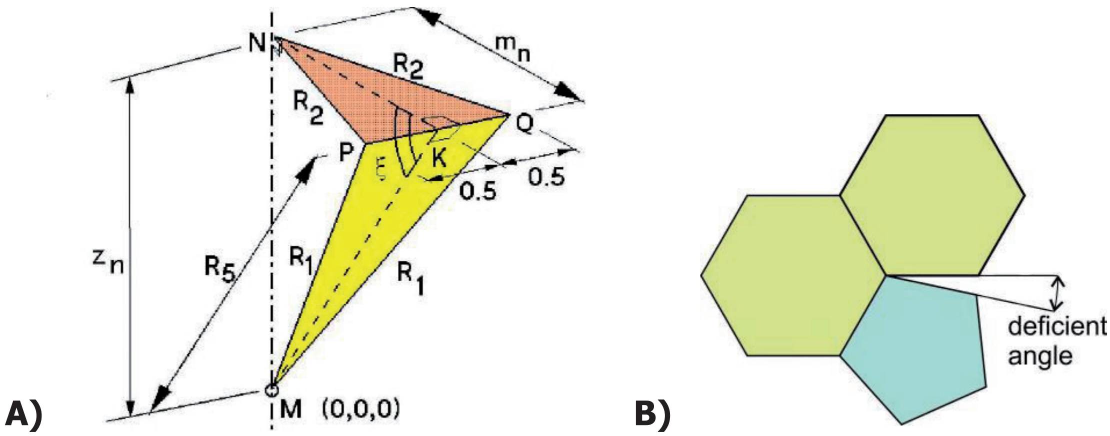


Fig. 1.17. *A) represents 1/n-th part of a polyhedral pyramid, which has a polygon with n sides as its basis and M (the center of the polyhedron) as its summit. MPKN is also called orthoscheme or quadrirectangular tetrahedron [1.5]. P and Q are the two ends and K the mid-point of a side. N is the center of the polygon and it has the distance z_n from the polyhedron center M. B) shows what is meant with 'deficient angle'.*

The following table 1.6 gives a review of the most relevant values, derived in the foregoing, including the total angles of the polygons that meet in a corner, and also the deficient angle (Fig. 1.17B) which represents the missing part that must be cut from the flat plane in order to give the polyhedron its spatial form.

P	R_1	R_3	R_5	R_6	Total angle	Deficient angle
1	0.61237244	0.57735027	0.35355339	0.20412415	180°	180°
2	0.86602540	0.81649658	0.70710678	0.57735027	270°	90°
3	0.70710678	0.70710678	0.50000000	0.35355339	240°	120°
4	1.40125854	0.93417236	1.30901699	1.22284749	324°	36°
5	0.95105652	0.85065081	0.80901699	0.68819096	300°	60°
6	1.17260394	0.90453403	1.06066017	0.95940322	300°	60°
7	1.00000000	0.86602540	0.86602540	0.75000000	300°	60°
8	1.58113883	0.94868330	1.50000000	1.42302495	330°	30°
9	1.77882365	0.95968298	1.70710678	1.63828133	330°	30°
10	1.39896633	0.93394883	1.30656296	1.22026295	330°	30°
11	1.31761091	0.97645098	1.26303344	1.20974121	345°	15°
12	1.61803399	0.95105652	1.53884177	1.46352549	336°	24°
13	1.47801866	0.97943209	1.42705098	1.37713161	348°	12°
14	1.96944902	0.98572192	1.92705098	1.88525831	348°	12°
15	1.34371337	0.92819138	1.24722317	1.15766179	330°	30°
16	1.23295051	0.97460776	1.17625090	1.12099102	348°	12°
17	3.80239450	0.99131669	3.76937713	3.73664646	354°	6°
18	1.15583738	0.97273285	1.09705384	1.03987315	348°	12°

Table 1.6. *Characteristic radii and angles of the uniform solids*

1.8. The dihedral angles

P	n_1	ξ_1	n_2	ξ_2	n_3	ξ_3
1	3)	35°15"51,8029'				
2	4)	45° 00"00,0000'				
3	3)	54°44"08,1971'				
4	5)	58°16"57,0921'				
5	3)	69°05"41,4332'				
6	3)	74°12"24,5914'	6)	35°15"51,8029'		
7	3)	70°31"43,6057'	4)	54°44"08,1971'		
8	4)	70°31"43,6057'	6)	54°44"08,1971'		
9	3)	80°15"51,8029'	8)	44°59"60,0000'		
10	3)	77°14"08,1971'	4)	67°29"60,0000'		
11	4)	77°14"08,1971'	6)	67°29"60,0000'	8)	57°45"51,8029'
12	3)	79°11"15,6589'	5)	63°26"05,8158'		
13	5)	73°31"40,0415'	6)	69°05"41,4332'		
14	3)	84°20"24,3826'	10)	58°16"57,0921'		
15	3)	76°37"02,2579'	4)	66°21"58,0904'		
16	3)	82°22"38,5253'	4)	76°43"02,9079'	5)	71°33"54,1842'
17	4)	82°22"38,5253'	6)	76°43"02,9079'	10)	65°54"18,5668'
18	3)	82°05"15,6589'	5)	70°50"32,0541'		

Table 1.7. *Dihedral angles between face and plane through edge PQ and centre M*

P	Dihedral angle 1		Dihedral angle 2		Dihedral angle 3	
1	(3,3)	70°31″43.6057′				
2	(4,4)	90°00″00.0000′				
3	(3,3)	109°28″16.3943′				
4	(5,5)	116°33″54.1842′				
5	(3,3)	138°11″21.8664′				
6	(3,6)	109°28″16.3943′	(6,6)	141°03″27.2114′		
7	(3,4)	125°15″51.8029′				
8	(4,6)	125°15″51.8029′	(6,6)	109°28″16.3943′		
9	(3,8)	125°15″51.8029′	(8,8)	90°00″00.0000′		
10	(3,4)	144°44″08.1971′	(4,4)	135°00″00.0000′		
11	(4,6)	144°44″08.1971′	(4,8)	135°00″00.0000′	(6,8)	125°15″51.8029′
12	(3,5)	142°37″21.4747′				
13	(5,6)	142°37″21.4747′	(6,6)	138°11″21.8664′		
14	(3,10)	142°37″21.4747′	(10,10)	116°33″54.1842′		
15	(3,3)	153°14″04.5158′	(3,4)	142°59″00.3483′		
16	(3,4)	159°05″41.4332′	(4,5)	148°16″57.0921′		
17	(4,6)	159°05″41.4332′	(4,10)	148°16″57.0921′	(6,10)	142°37″21.4747′
18	(3,3)	164°10″31.3178′	(3,5)	152°55″47.7130′		

Table 1.8. *The dihedral angles between all meeting pairs of polygons. The numbers of their sides are indicated in brackets.*

The dihedral angle between the n-gon and the centri-plane PQM of Fig. 1.8B:

$$\xi_n = \arctan \frac{z_n}{m_n} \qquad \{1.16\}$$

m_n follows from equation {1.3}
At any edge of a polyhedron always two polygons meet. The total dihedral angle is therefore composed of two parts, each of which is defined by its adjacent polygon.

1.9. Areas of the polyhedra

The total area of a polyhedron is found by the summation of the areas of all polygons, occurring in this polyhedron.
Area $P_{TOT} = \Sigma\ (q_{n1}AreaP_{n1} + q_{n2}\ AreaP_{n2}\ + q_{n3}AreaP_{n3})$ {1.17}

1.10. Volumes of the polyhedra

The volume of a polyhedron can be found by adding the volumes of all centri-pyramids. The volume of such a pyramid:
Vol $P_n = {}^1/_3$ * area of polygon * height $= {}^1/_3 * {}^1/_2 * {}^1/_2 * m_n * z_n * 2\ n$
$= {}^1/_6 * m_n * z_n * n$ {1.18}

The total volume of the polyhedron:
Vol $P_{TOT} = \Sigma\ (q_{n1}VolP_{n1} + q_{n2}\ VolP_{n2}\ + q_{n3}VolP_{n3})$ {1.19}

A summary of the values for the areas and volumes is given in the following table together with those of the reciprocals, the derivation of which is explained in the next paragraph.

P	$VolumeP_{tot}$	$AreaP_{tot}$	$VolumeR_{tot}$	$AreaR_{tot}$
1	0.11785113	1.73205080	0.11785113	1.73205080
2	1.00000000	6.00000000	1.33333333	6.92820323
3	0.47140452	3.46410161	0.35355339	3.00000000
4	7.66311896	20.64572880	9.24180829	21.67283942
5	1.18169499	8.66025403	1.80901699	7.88596668
6	1.71057599	11.12435565	5.72756493	17.90977386
7	1.35702260	9.46410161	1.38648539	9.54594154
8	11.31370850	26.78460969	14.31891232	30.18691769
9	13.59966329	31.43466436	23.31370850	41.69176749
10	8.71404521	21.46410161	8.75069057	21.51345464
11	41.79898987	61.75517243	49.66382185	67.42484815
12	13.83552594	29.30598284	14.80021243	30.33813728
13	55.28773076	71.60725303	59.87641488	75.56554470
14	85.03966456	100.99076015	111.14946533	115.56968557
15	7.88947740	19.85640646	7.44739519	19.29940656
16	41.61532378	59.30598284	41.25536942	59.76739510
17	206.80339887	174.29203034	228.17899489	183.19554518
18	37.61664996	55.28674495	37.58842367	55.28053092

Table 1.9. *The areas and volumes of polyhedra and of their reciprocals*

1.11. Prisms and antiprisms

There are numbers of other figures that also answer the definition that was given in paragraph 1.1, namely some of the prisms and antiprisms. In fact they form endless rows, the prisms having two parallel polygonal sides and a mantle of squares, where the antiprisms instead have a mantle of equilateral triangles. Following definition 1) only those are considered to be uniform polyhedra, that have parallel sides with 3, 4, 5, 6, 8, or 10 edges.

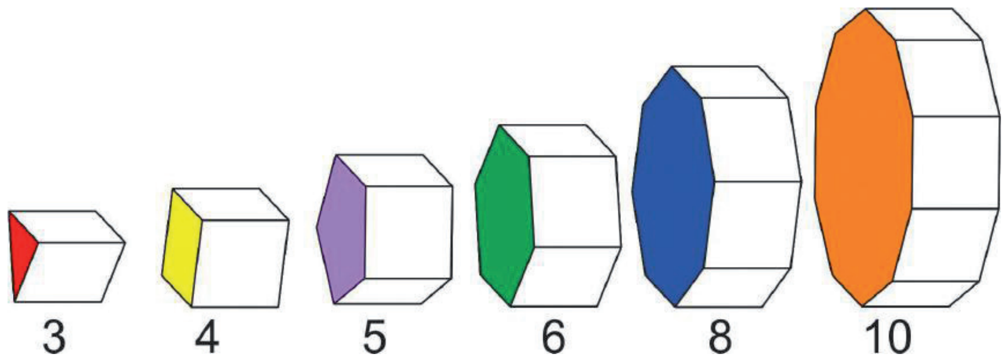


Fig.1.18. *Row of the uniform prisms, following the definition in paragraph 1.1.*

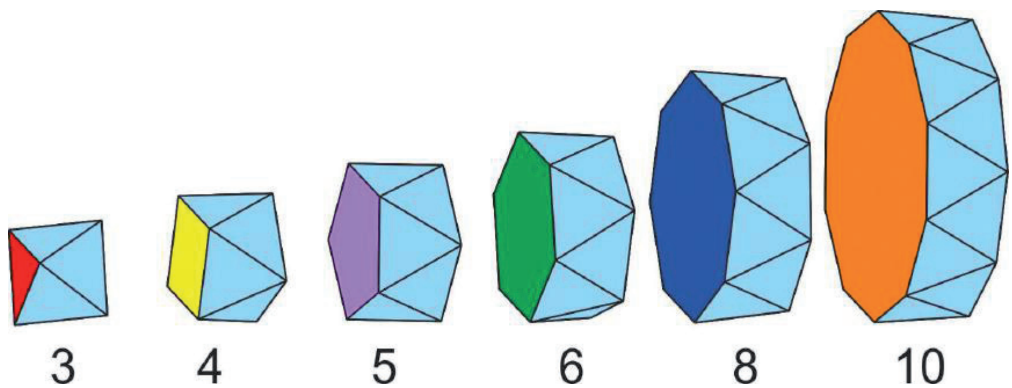


Fig. 1.19. *The uniform antiprisms*

These solids have similar characteristics as the Archmedean polyhedra, in this case consisting of two kinds of polygons. They also have similar vertex figures as in Fig. 1.20. In Chapter 6 an overview is given of the whole group of prismatics, or prism based figures and structural forms. They will be treated in this Chapter 6 separately and following a more general approach.

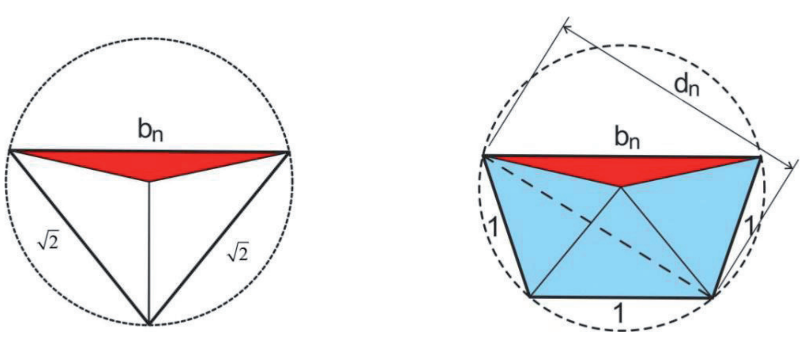


Fig. 1.20. *Vertex figures of the prisms and antiprisms*.

1.12. Isomeres

Four members of the semi-regular polyhedra allow different arrangements of parts of them that in fact answer most of the definitions of the uniform convex polyhedra on page 1.1, but with the exception of point 3, which says that all vertices are identical. The four solids in Fig. 1.21 have parts, which can be turned with respect to the rest of the solid. P7, P10 and P12 have two possible variants, whereas P16 has maybe five possible different arrangements.These are called Isomeres.

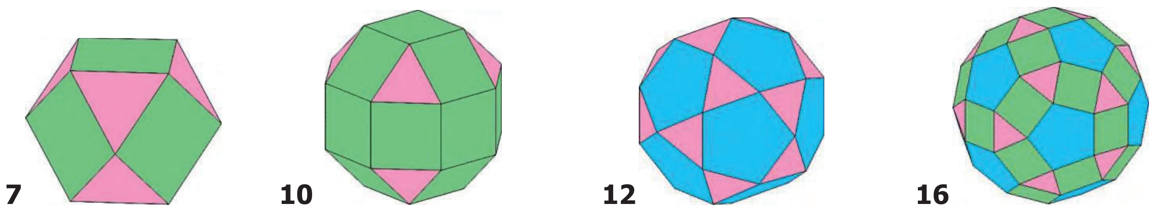


Fig. 1.21. *The four Archimedeans, that allow different arrangements*

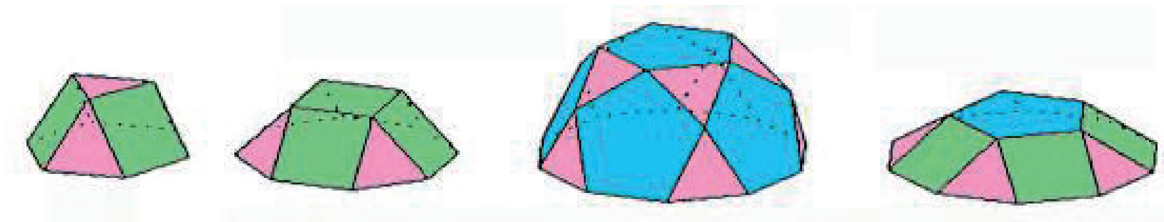


Fig. 1.22. *The pyramidal caps that can be turned around.*

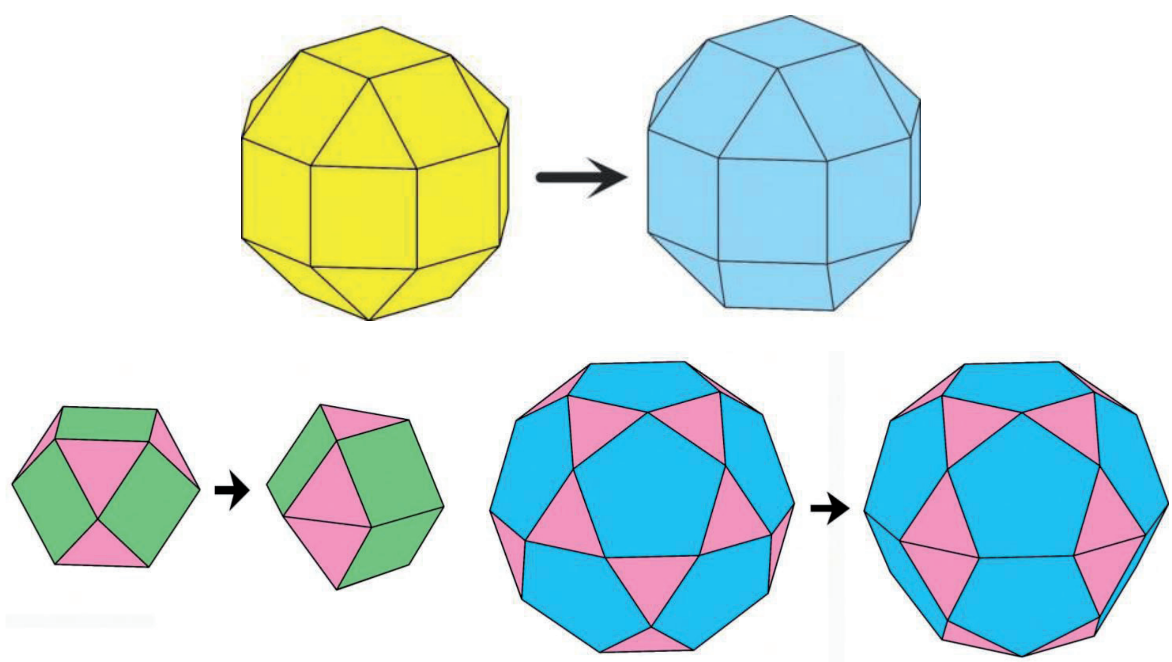


Fig. 1.23. *Some isomeric forms*

1.13. Pyramidization

On each polygonal plane of any polyhedron shallow pyramid can be erected, of which the apex, just like all corners of the polygon of the polyhedron, all lie on the circumscribed sphere. We have introduced here the term 'pyramidization'.

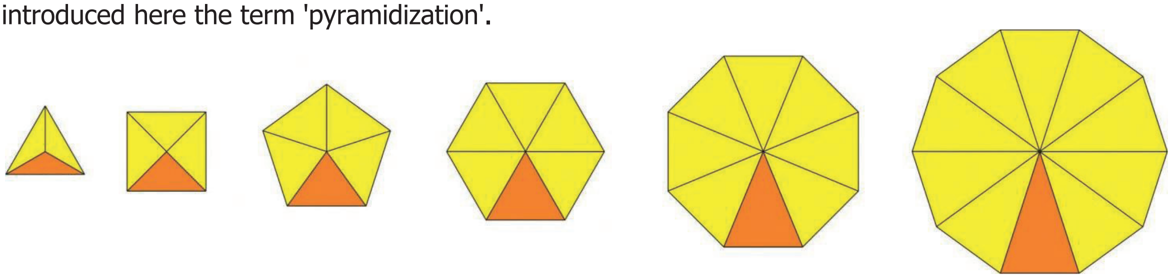


Fig. 1.24. *The six pyramidized polygons, shown in plan.*

Height of the pyramid: $h_n = R_1 - z_n = R_1 - \sqrt{R_1^2 - R_2^2}$ {1.20}

Length of inclined edge: $e_n = \sqrt{h_n^2 + R_2^2}$ {1.21}

Height of isosceles triangle: $h_a = \sqrt{e_n^2 - 0.25}$ {1.22}

Basis angle of triangle: $\lambda = \arctan\left(2\sqrt{e_n^2 - 0.25}\right)$ {1.23}

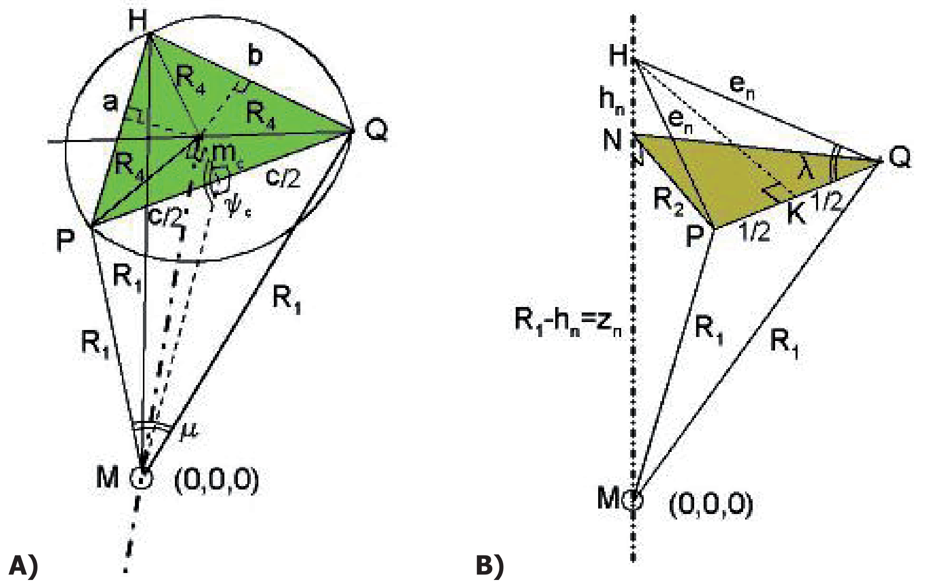


Fig. 1.25. A) Pyramidization of polygon sector, B) Characteristic aspects of triangle on sphere

For the determination of the dihedral angles along the edges of the triangular sides of this pyramid a general approach can be used, where the corners of a triangle with the sides a, b and c lie on the sphere with the radius R_1. Around this triangle a circle can be circumscribed, of which the radius is called here R_4 with N as the centre of this circle.

$$m_c = \sqrt{R_4^2 - \left(\frac{c}{2}\right)^2} = \sqrt{R_4^2 - 0.25c^2}$$

$$y = \sqrt{R_1^2 - R_4^2}$$ {1.24}

$$\psi_a = \arctan\left(\frac{y}{m_a}\right),\ \psi_b = \arctan\left(\frac{y}{m_b}\right) \text{ and } \psi_c = \arctan\left(\frac{y}{m_c}\right)$$

In the shallow pyramid the triangle is isosceles and the sides are e_n and 1.

Half the sum of the sides: $S = \frac{2e_n + 1}{2} = e_n + 0.5$

Area of the triangle: $O = \sqrt{(e_n + 0.5)(0.5)^2(e_n - 0.5)} = 0.5\sqrt{e_n^2 - 0.25}$

Radius of circumscribed circle: $R_4 = \frac{e_n}{4O}\ \psi_1 = \arctan\left(\frac{y}{m_1}\right) = \frac{e_n^2}{\sqrt{4e_n^2 - 1}}$

Two different dihedral angles occur:

1) On an inclined edge:

$$m_e = \sqrt{R_4^2 - \left(\frac{e_n}{2}\right)^2} = \sqrt{R_4^2 - 0.25e_n^2} \qquad \{1.25\}$$

$$\psi_e = \arctan\left(\frac{y}{m_e}\right) \qquad \{1.26\}$$

2) On the edge with length=1:

$$m_1 = \sqrt{R_4^2 - 0.25} \qquad \{1.27\}$$

$$\psi_1 = \arctan\left(\frac{y}{m_1}\right) \qquad \{1.28\}$$

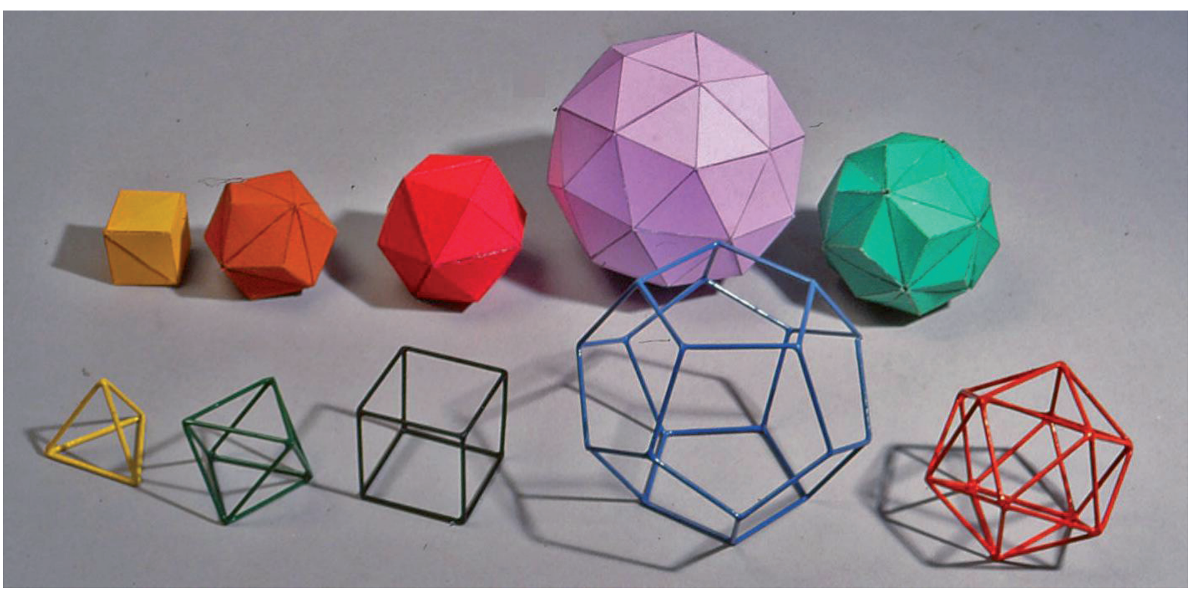


Fig. 1.26. *Models of pyramidized Platonic polyhedra; it is interesting to see that the tetrahedron converts into a cube.*

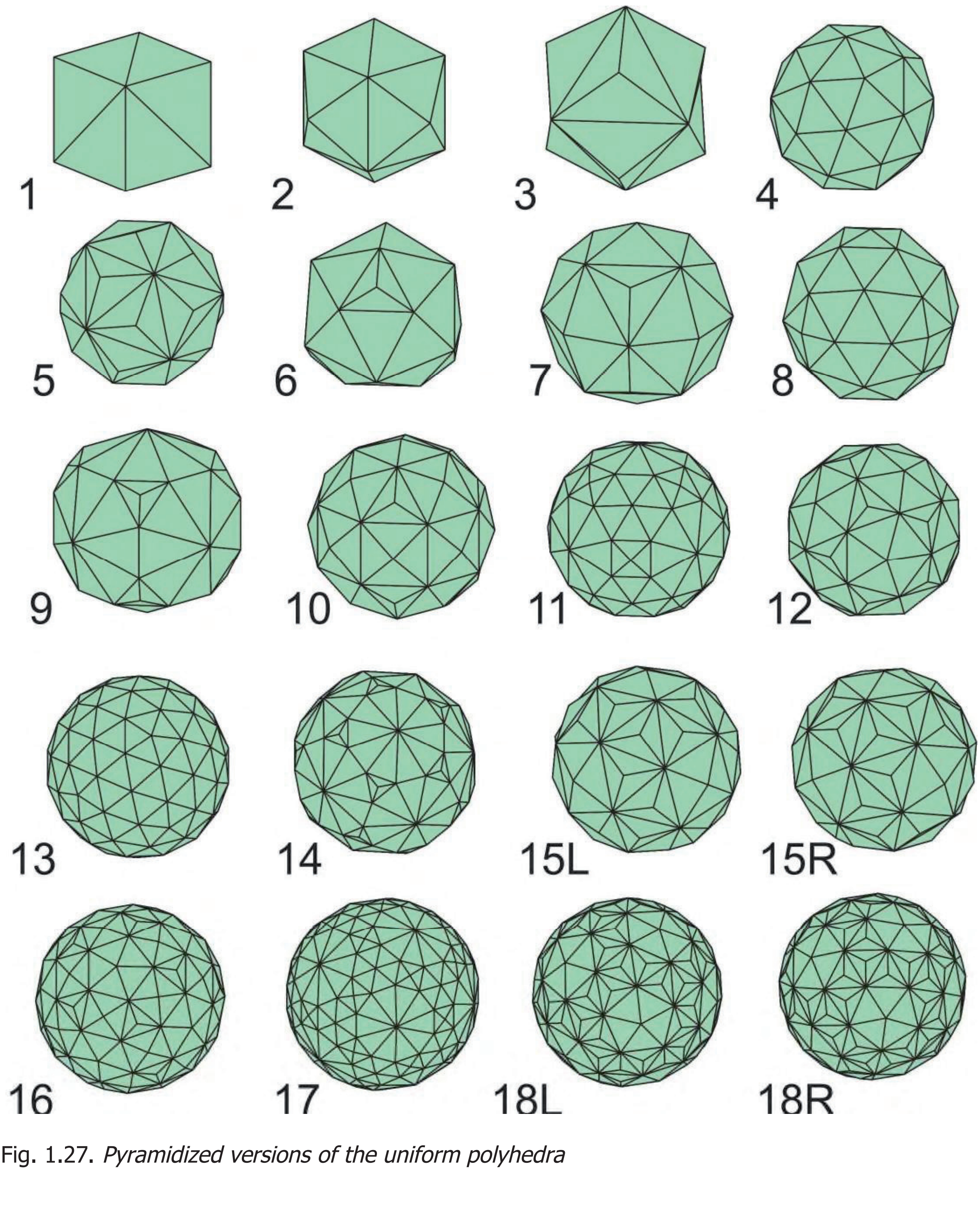


Fig. 1.27. *Pyramidized versions of the uniform polyhedra*

1.14. Deltahedra

A class of figures, of which all faces are regular triangles, is called that of Deltahedra. Only eight of these are convex. The most obvious of these are of course the three triangular Platonic solids: the Tetrahedron, the Octahedron and the Icosahedron (P1, P3 and P5). The Octahedron can be seen as two square pyramids that are posed opposite to each other against their common square faces. There are two more of these: the triangular pyramid (enumerated in Fig. 1.28 as D2) and the one composed of two pentagonal pyramids, D4.

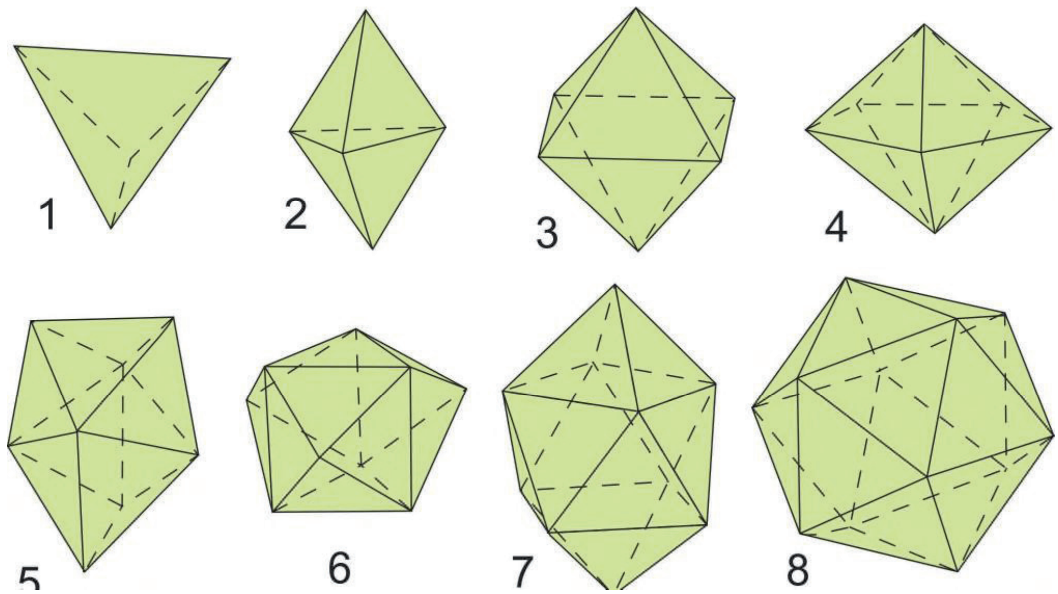


Fig. 1.28. *The eight convex deltahedra*

These different forms are called after their numbers of faces:

D1. Tetradeltahedron (4 triangles): more familiar as the tetrahedron.
D2. Hexadeltahedron (6 triangles): a double tetrahedron.
D3. Octadeltahedron (8 triangles): the octahedron.
D4. Dekadeltahedron (10 triangles): a double pentagonal pyramid (icosahedron cap).
D5. Dodekadeltahedron (12 triangles).
D6. Tetrakaidekadeltahedron (14 triangles): a triangular prism with pyramids on its square faces.
D7. Hexakaidekadeltahedron (16 triangles): a 4-sided antiprism with pyramids on its square faces.
D8. Icosadeltahedron (20 triangles), synonymus to the regular Icosahedron but also similar to two pentagonal pyramids, placed on the parallel sides of a pentagonal antiprism.

1.15. References

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Chapter 2. THE INDIVIDUAL POLYHEDRA

The total number of the regular polyhedra is 5 and not more than 5. This can easily be understood from table 1.5, where the deficient angles of all uniform poyhedra are given (See also Fig. 15.7). The total sum of the top angles of the polygons that meet in a vertex, must always be less than 360° in order to give the polyhedron its round form. If this angle is equal to 360° the plane becomes flat and if its larger, too much material is available so that the figure becomes wrinkled. Following the conditions in Chapter 1 we only use the polygons with 3, 4, 5, 6, 8, and 10 sides. In each vertex at least three polygons must meet in order to form a space angle.

In the previous Chapter 1 an algebraic approach was followed to obtain the geometric data of the polyhedra. In Chapter 14 all numeric data are given following this approach. But in special cases it is often desirable to have a formula available to calculate a particular property as exactly as possible. M. Brückner derived many of these and these are given in the following chapter, and in more concise form in the second part of Chapter 14. These formulae are given in coherence with the respective polyhedra and their net, if folded out.
Some of the data derived by Brückner were corrected, if necessary and if indicated, with the help of Chapter 14 and computed earlier by Huybers in 1976 [2.2].

2.1. Tetrahedron P1

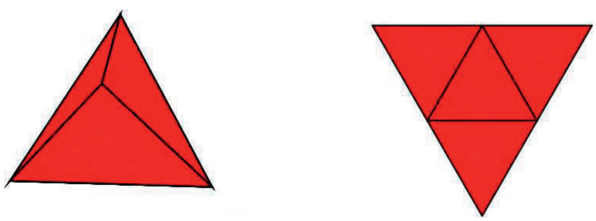

Fig. 2.1. *The Tetrahedron and its net*

a = edge length of the n-gons

Dihedral angle between the 3-gon and the centri-plane through the unit edge:
$$\xi_3 = \arccos\left(\frac{1}{3}\right) = 35^0 15'36.8''$$

Radius of the circumscribed sphere: $R_1 = \frac{a}{4}\sqrt{6}$

Total surface area of the polyhedron: $A = a^2\sqrt{3}$

Total volume of the polyhedron: $V = \frac{a^2}{12}\sqrt{2}$

Distance of a 3-gon from the centre of the circumscribed sphere: $z_3 = \frac{a}{12}\sqrt{6}$

2.2. Cube P2

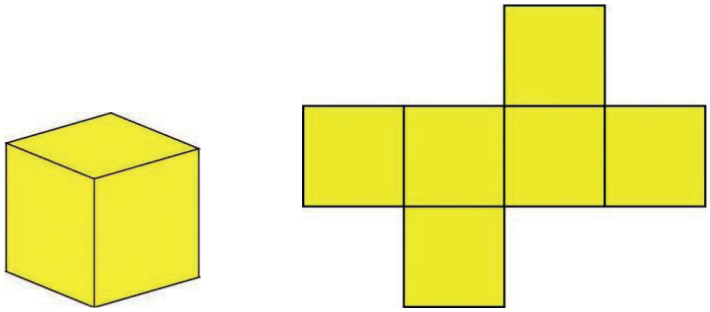

Fig. 2.2. *The Cube and its net*

Dihedral angle between the 4-gon and the centri-plane through the unit edge:
$\xi_4 = 45^0$

Radius of the circumscribed sphere: $R_1 = \frac{a}{2}\sqrt{3}$

Total surface area of the polyhedron: $A = 6a^2$
Total volume of the polyhedron: $V = a^3$

Distance of a 4-gon from the centre of the circumscribed sphere: $z_4 = \frac{a}{2}$

2.3. Octahedron P3

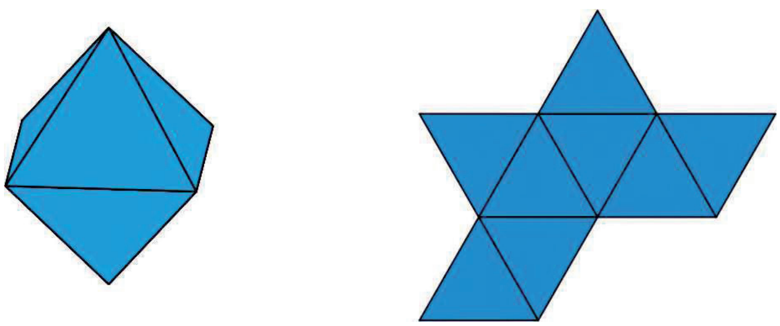

Fig. 2.3. *The Octahedron and its net*

Dihedral angle between the 3-gon and the centri-plane passing through the unit edge:
$\xi_3 = \arctan(\sqrt{2}) = 54^0 29'8.2''$

Radius of the circumscribed sphere: $R_1 = \frac{a}{2}\sqrt{2}$

Total surface area of the polyhedron: $A = 2a^2\sqrt{3}$

Total volume of the polyhedron: $V = \frac{a^2}{3}\sqrt{2}$

Distance of a 3-gon from the centre of the circumscribed sphere: $z_3 = \frac{a}{6}\sqrt{6}$

2.4. Dodecahedron P4

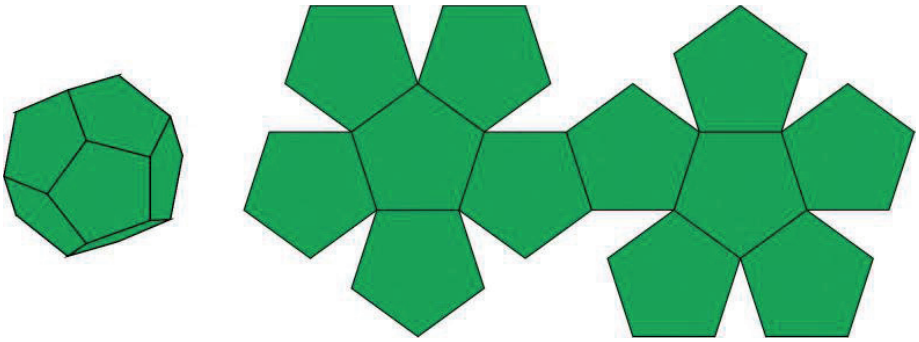

Fig. 2.4. *The Dodecahedron and its net*

Dihedral angle between the 5-gon and the centri-plane through the unit edge:

$$\xi_{3,5} = \arccos(-\sqrt{\frac{1}{5}}) = 58^0 16'42.1"$$

Radius of the circumscribed sphere: $R_1 = \frac{a}{4}\sqrt{18+6\sqrt{5}}$

Total surface area of the polyhedron: $A = 3a^2\sqrt{25+10\sqrt{5}}$

Total volume of the polyhedron: $V = \frac{a^3}{4}\left(15+7\sqrt{5}\right)$

Distance of a 5-gon from the centre of the circumscribed sphere: $z_5 = \frac{a}{2}\sqrt{\frac{25+11\sqrt{5}}{10}}$

2.5. Icosahedron P5

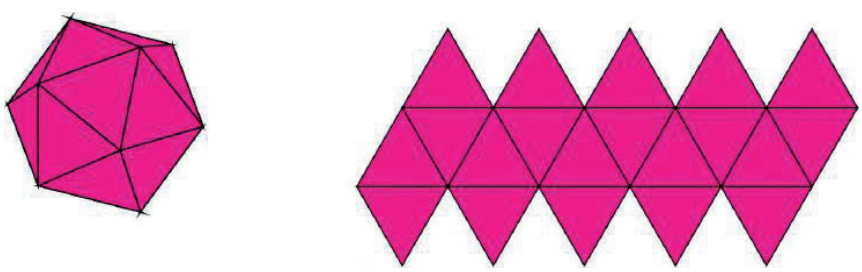

Fig. 2.5. *The Dodecahedron and its net*

Dihedral angle between the 3-gon and the centri-plane through the unit edge:

$$\xi_3 = \arcsin\left(\frac{\sqrt{15}+\sqrt{3}}{6}\right) = 69^0 16'42.1"$$

Radius of the circumscribed sphere: $R_1 = \frac{a}{4}\sqrt{10+2\sqrt{5}}$

Total surface area of the polyhedron: $A = 5a^2\sqrt{3}$

Total volume of the polyhedron: $V = \frac{a^3}{4}\left(15 + 7\sqrt{5}\right)$

Distance of the 3-gon from the centre of the circumscribed sphere: $z_3 = \frac{a}{12}\left(3 + \sqrt{5}\right)\sqrt{3}$

2.6. Truncated Tetrahedron P6

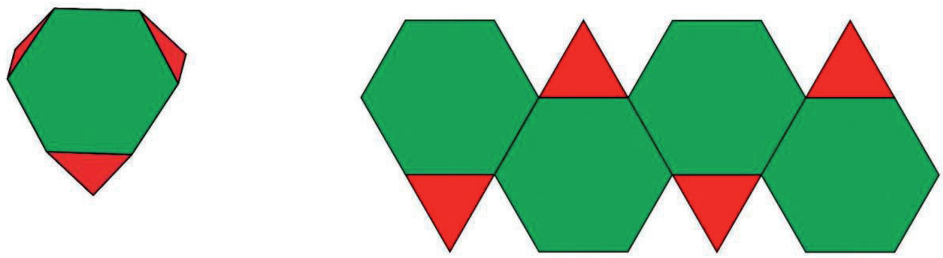

Fig. 2.6. *The Truncated Tetrahedron and its net*

Dihedral angle between a 3-gon and the centri-plane through the unit edge: $\xi_3 = \arctan\left(\frac{5}{2}\sqrt{2}\right)$

Dihedral angle between a 6-gon and the centri-plane through the unit edge: $\xi_6 = \arctan\left(\frac{1}{2}\sqrt{2}\right)$

Radius of the circumscribed sphere: $R_1 = \frac{a}{2}\sqrt{22}$

Total surface area of the polyhedron: $A = 7a^2\sqrt{3}$

Total volume of the polyhedron: $V = \frac{23}{12}a^3\sqrt{2}$

Distance of a 3-gon from the centre of the circumscribed sphere: $z_3 = \frac{5}{12}a\sqrt{6}$

Distance of a 6-gon from the centre of the circumscribed sphere: $z_6 = \frac{a}{4}\sqrt{6}$